SATstuff#4

This week we'll review functions and properties of exponents.

FUNCTIONS

A FUNCTION is a rule that assigns to each input exactly one corresponding output.
You can think of a function as 'acting on' an input and producing an output.
Using normal function notation, if f is the name of the rule, and x is the input, then $f(x)$ (pronounced as "f of x") is the corresponding output.

So, what is $g(3)$? It is the output from the function g, when the input is 3.
What is $f(x+h)$? It is the output from the function f, when the input is $x+h$.

The SAT use more creative notation to illustrate the process of taking number(s), doing something to them, and getting a unique output.

For example, they might define $\boxed{\text{x y}}$ to mean $x+3y$.

Then, $\boxed{2\ \ 5}$ represents $2+3\cdot 5=17$.

Verbalizing functions (rules) as SEQUENCES OF OPERATIONS
The best way to think of any function rule is as a sequence of operations.
For example, $2x+3$ represents the rule: take a number, multiply by 2, then add 3.
The rule $3x^2$ represents the rule: take a number, square it, then multiply by 3.
The rule $(3x)^2$ represents the rule: take a number, multiply by 3, then square the result.

TRY THESE:

(a) In words, what does the function notation $f(3)$ represent?

(b) Put the rule $f(x)=5x-1$ into words: take a number, ...

(c) Suppose that $\boxed{\text{x y}}$ means mean $2x+y$.
Find $\boxed{1\ \ 3}$.

(d) Use a mathematical expression to represent this rule: take a number, multiply it by 5, then subtract 3.

If you're shooting for a 600 on the Math SAT, then you should work at the pace of about one minute, thirty seconds per multiple-choice problem.
Get a feel for this pace by trying the following function problems:

1. For all positive numbers, $\Rightarrow x$ represents the nearest even integer greater than x.
If $x = 5$, then $\Rightarrow x$ is

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

2. Let x be any positive integer. The operation $*$ is defined in the following way:
x^* represents the least prime number greater than x.
If $x = 18$, then $x^* =$

(A) 15 (B) 17 (C) 19 (D) 20 (E) 23

3. Let a "k-triple" be defined as $(\frac{k}{2}, k, \frac{3}{2}k)$ for some number k.
Which of the following is a k-triple?

(A) $(0, 5, 10)$ (B) $(4\frac{1}{2}, 5, 6\frac{1}{2})$ (C) $(25, 50, 75)$ (D) $(250, 500, 1000)$ (E) $(450, 500, 650)$

4. If $\boxed{\begin{matrix} w & x \\ y & z \end{matrix}}$ is defined to equal $wy - xz$, and $\boxed{\begin{matrix} w & x \\ y & z \end{matrix}} - K = 0$, then $K =$

(A) $wy - wz$ (B) $xz + wy$ (C) $-xz$ (D) $xz - wy$ (E) $wy - xz$

The next two questions refer to the following definition:

$\boxed{\begin{matrix} W & X \\ Y & Z \end{matrix}}$ is a *number square* if $W + Z = X + Y$ and $2W = 3X$.

5. If $\boxed{\begin{matrix} 3 & X \\ Y & 5 \end{matrix}}$ is a *number square*, then what is the value of Y?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

6. If $\boxed{\begin{matrix} W & X \\ Y & W \end{matrix}}$ is a *number square*, then $Y =$

(A) $\frac{3}{4}W$ (B) W (C) $\frac{4}{3}W$ (D) $3W$ (E) $4W$

7. Let $\#$ be defined by $z \# w = z^w$.
If $x = 5 \# a$, $y = 5 \# b$, and $a + b = 3$, then what is the value of xy?

(A) 15 (B) 30 (C) 75 (D) 125 (E) 243

EXPONENT LAWS:

Exponent notation is a shorthand for repeated multiplication:

$$x^3 \quad \text{means} \quad x \cdot x \cdot x$$

$$(a+b)^2 \quad \text{means} \quad (a+b)(a+b) = a^2 + 2ab + b^2 \quad \text{(use FOIL)}$$

Here are the basic laws for working with exponents:

Things multiplied, same base, ADD the exponents:
$x^n x^m = x^{m+n}$
Example: $(2^3)(2^5) = (2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = 2^{3+5} = 2^8$

Things divided, same base, SUBTRACT the exponents:
$\dfrac{x^m}{x^n} = x^{m-n}$
Example: $\dfrac{x^5}{x^3} = \dfrac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = x^{5-3} = x^2$

Something to a power, to a power, multiply the exponents:
$(x^m)^n = x^{mn}$
Example: $(x^2)^3 = (x^2)(x^2)(x^2) = (x \cdot x)(x \cdot x)(x \cdot x) = x^{2 \cdot 3} = x^6$

Trade a negative exponent in for a "flip":
$x^{-n} = \dfrac{1}{x^n}$ OR $\dfrac{1}{x^{-n}} = x^n$
Example: $(a+b)^{-2} = \dfrac{1}{(a+b)^2}$

Fractional exponents—the denominator tells the kind of root;
the numerator is a power, which can go inside or outside:
$x^{a/b} = (x^{1/b})^a = (\sqrt[b]{x})^a$ (usually, this name is easiest)
$x^{a/b} = (x^a)^{1/b} = \sqrt[b]{x^a}$
Example: $8^{5/3} = (\sqrt[3]{8})^5 = 2^5 = 32$

In particular, $x^{1/2} = \sqrt{x}$.
Recall: $\sqrt{x}$ is the nonnegative number which, when squared, gives x:
Example: $\sqrt{4} = 2$, even though both $2^2 = 4$ and $(-2)^2 = 4$.
You can't take EVEN roots of NEGATIVE numbers: $\sqrt{-4}$ is not defined.

SPECIAL CASES THAT YOU COME UP A LOT:
$\sqrt{xy} = \sqrt{x}\sqrt{y}$ (only when x and y are BOTH positive)
$\sqrt{\dfrac{x}{y}} = \dfrac{\sqrt{x}}{\sqrt{y}}$ (only when x and y are BOTH positive)

TRY THESE:
(a) $\sqrt{(-2)(-2)} =$

(b) $\sqrt{\dfrac{9}{100}} =$

(c) $\sqrt{x^2} =$ (Be careful!)

(d) $(27)^{2/3} =$ (Do this WITHOUT a calculator.)

Now, you try these:

1. If $2^y = 8$ and $y = \frac{x}{2}$, then $x =$

(A) 6 (B) 5 (C) 4 (D) 3 (E) 2

2. If $x = 5^y$ and $y = z + 1$, then what is $\frac{x}{5}$ in terms of z?

(A) z (B) $z + 1$ (C) 5^z (D) 5^{z+1} (E) 5^{z+1}

3. If $x + 1 = 7$, then $(x + 2)^2 =$

(A) 25 (B) 36 (C) 49 (D) 64 (E) 81

4. If $\left(x + \dfrac{1}{x}\right)^2 = 25$, then $\dfrac{1}{x^2} + x^2 =$

(A) 23 (B) 24 (C) 25 (D) 27 (E) 624

5. If $(5^3)(2^5) = 4(10^k)$, then $k =$

(A) 2 (B) 3 (C) 4 (D) 6 (E) 8

6. $\left[(2x^2y^3)^2\right]^3 =$

(A) $4x^4y^6$ (B) $12x^4y^6$ (C) $64x^4y^6$ (D) $64x^{12}y^{18}$ (E) $64x^{64}y^{216}$

7. $a \cdot 3 \cdot b^2 \cdot \dfrac{1}{2} =$

(A) a^3b (B) $1.5ab^2$ (C) $1.5a^2b^2$ (D) $3ab$ (E) $6ab^2$

EXTRA PROBLEMS:

1. For any sentence J, the expression $N_t(J)$ is defined to mean the number of times the letter "t" appears in J. If J is the sentence "All cats are good luck," then $N_t(J) =$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Questions (2) and (3) refer to the following definition:

$< x >$ is defined as 1 less than the number of digits in the integer x.

For example, $< 100 >= 3 - 1 = 2$.

2. If x is a positive integer less than $1,000,001$, then $< x >$ is at most

(A) 5 (B) 6 (C) 7 (D) 999,999 (E) 1,000,000

3. If x has $1,001$ digits, then what is the value of $<<< x >>>$?

(A) 997 (B) 1 (C) 0 (D) -1 (E) It cannot be determined from the information given.

4. For all numbers x, y, and z, if the operation ϕ is defined by the equation $x \ \phi \ y = x + xy$, then $x \ \phi \ (y \ \phi \ z) =$

(A) $x + xy + xyz$ (B) $x + xyz$ (C) $x + xy + z + xz$ (D) $x + y + yz$ (E) $x + y + xyz$

Questions (5) and (6) refer to the following definition:

$$= \frac{a \cdot b}{c} + \frac{b \cdot c}{a} + \frac{c \cdot a}{b} \text{ for all nonzero } a, b, \text{ and } c.$$

For example,

$$= \frac{2 \cdot 4}{6} + \frac{4 \cdot 6}{2} + \frac{6 \cdot 2}{4} = \frac{4}{3} + 12 + 3 = 16\frac{1}{3}$$

5. $=$

(A) 1 (B) 9 (C) 10 (D) 16 (E) 26

6. If $x \neq 0$, $=$

(A) $x^6 + x^4 + x^2$ (B) $x^5 + x + \frac{1}{x}$ (C) $x^4 + x^3 + 1$ (D) $x^4 + x^2 + 1$ (E) $x^2 + x + 1$