SATstuff#2

Problems involving remainders are popular on the SAT test.

REMAINDERS: Suppose you have 20 apples and want to make piles of size 7. You can make 2 piles, with 6 left over. The number left over (*remaining*) is called the **remainder**. Numbers have lots of different names! The name $20 = 2 \cdot 7 + 6$ shows that you can get 2 piles of 7, with 6 left over.

Your calculator can help you find remainders. To find the remainder when 20 is divided by 7, do this:

- How many piles of 7 can I get from 20? Key in $20/7$, giving something like 2.857142857. You can get 2 full piles. (Throw away the stuff after the decimal point.)
- Then, $20 - 2 \cdot 7 = 20 - 14 = 6$ gives the remainder.

The key to remainders is making as many piles of a given size as possible, and seeing what is left over.

TRY THESE:

(a) What is the remainder when 20 is divided by 6? (Do this without your calculator.)

(b) What is the remainder when 217 is divided by 11? (Use your calculator)

(c) If a number goes into n evenly, then what is the remainder?

Here's a typical problem.

Try it yourself first (on this page).

You'll get 1.25 minutes—75 seconds—which is the average time for a multiple-choice problem (if you do them all) on the SAT.

Then, three different solution approaches are discussed.

Finally, you'll apply the concept to several practice problems.

When n is divided by 4, the remainder is 2. What is the remainder when $6n$ is divided by 4?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Note: When you divide by 4, you'll never get a remainder of 4—because if you have 4 left over, then you can make another pile of 4! So, the only possible remainders are 0, 1, 2, and 3. You should eliminate (E) immediately.

PICTURE SOLUTION: (For people who like pictures.) Draw a picture representing n: piles (of size 4) and what's left over. Repeat this picture 6 times. (Of course, once you do this a couple times, you won't really have to draw the whole picture.) Focus on what's left over; make piles of size 4. What's left over? None! The answer is (a).

PICK A NUMBER: (For people who like working with actual numbers.) We know that when n is divided by 4, the remainder is 2. What numbers could n be?

Start with the remainder only (no full piles) and then keep adding on piles of size 4:

$n = 2,\ 6,\ 10,\ 14,\ 18,\ \ldots$.

Pick one of the simplest numbers that n could be. For example, choose $n = 10$.

Now, what is $6n$? Well, $6n = 6 \cdot 10 = 60$, which is 15 piles of 4 with none left over. The answer is (a).

(Pick other values of n and see that you still get the same answer.)

ALGEBRAIC SOLUTION: (For algebra lovers, like Dr. Fisher, this is the best solution.) When n is divided by 4, the remainder is 2. So, $n = 4p + 2$. (n can be made into p piles of 4, with 2 left over.)

How many piles of 4 in $6n$?

$$\begin{aligned} 6n &= 6(4p+2) \\ &= 6 \cdot 4p + 6 \cdot 2 \\ &= 4 \cdot 6p + 12 \\ &= (\text{piles of size } 4) + 12 \\ &= (\text{piles of size } 4) + (4+4+4) \end{aligned}$$

So, $6n$ can be completely broken into piles of size 4. None left over! The answer is (a).

Note that we really only needed to deal with the $6 \cdot 2 = 12$ part—why?

PRACTICE PROBLEMS:

1. When x is divided by 4, the remainder is 3. What is the remainder when $7x$ is divided by 4?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

2. When n is divided by 11, the remainder is 7. What is the remainder when $5n$ is divided by 11?

(A) 0 (B) 2 (C) 4 (D) 7 (E) 9

3. When n is divided by 5 the remainder is 2. When m is divided by 5 the remainder is 1. What is the remainder when $m + n$ is divided by 5?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

4. If $a = 4b + 26$, and b is a positive integer, then a could be divisible by all of the following EXCEPT

(A) 2 (B) 4 (C) 5 (D) 6 (E) 7

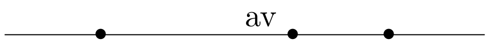
5. If the remainder when x is divided by 5 equals the remainder when x is divided by 4, then x could be any of the following EXCEPT

(A) 20 (B) 21 (C) 22 (D) 23 (E) 24

Problems involving averages are popular on the SAT test.

AVERAGES: To average n numbers, you add them all up and then divide by n. That is, the average of the n numbers $x_1, x_2, \ldots, x_n$ is given by the formula $\dfrac{x_1 + x_2 + \cdots + x_n}{n}$.

When you're working with averages, you should keep a "see-saw" in mind! The average gives the balancing point, as shown below: imagine a single pebble (all the same size) placed at each of the numbers being averaged. Where should the support go so that everything is in balance?



There are several results about averages that follow immediately from this "balancing" viewpoint.

- The average of two numbers is exactly halfway between the numbers.
- When you average numbers, the average is always between the greatest and least number. For example, suppose you're averaging the numbers -1, 1, -3, and 5. Then, the average must be between -3 and 5. (Check that the average is 0.5.)
- If you have an average, and you add in another number that's greater than the average, then the average must go up. Similarly, if you add in another number that's less than the average, then the average must go down.

Note: An AVERAGE is also called an ARITHMETIC MEAN.

TRY THESE:

(a) What is the average of 5 and 9? (Don't do any arithmetic—think of balancing points.)

(b) What is the average of 101, 103, and 105? (Don't do any arithmetic—think of balancing points.)

(c) What is the average of $x - b$, x, and $x + b$?

(d) What is the average of $x - 2$, $x - 1$, $x + 1$, and $x + 2$?

Here's a typical problem.

Try it yourself first (on this page).

If you're shooting for a 550 on the Math SAT, then you should only attempt about 14 out of 20 questions in a 25-minute multiple-choice section; that's about 1 minute 47 seconds per problem. [Cracking the SAT, 2009 edition, the Princeton Review, page 27]

Then, three different solution approaches are discussed.

Finally, you'll apply the concept to several practice problems.

The average weight of five packages is 20 pounds. Four of the packages weigh 5, 10, 12, and 22 pounds. What is the weight of the fifth package (in pounds)?

(A) 15 (B) 25 (C) 45 (D) 51 (E) 100

INTUITION/GUESS-AND-CHECK: A quick sketch and some good intuition about the "balancing point" will tell you that the answer CAN'T BE (A), (B), or even (E). Try the remaining two numbers. Averaging 5, 10, 12, 22, and 45 gives (use your calculator) 18.8, which is a bit too small. The correct answer is (D).

ALGEBRAIC SOLUTION: Let x be the unknown number. Then,

$$\begin{aligned} \frac{5+10+12+22+x}{5} &= 20 \\ 49+x &= 100 \\ x &= 51 \end{aligned}$$

The correct answer is (D).

ALTERNATE ALGEBRAIC SOLUTION: The sum of the distances from the average must be the same on both sides. Look at the sketch below: the 'pebbles' to the left give a combined distance of $15+10+8=33$. So, the 'pebbles' on the right must also sum to 33. Then, $2+x=33$ means that the remaining distance from the average must be 31. So, the last number is $20+31=51$.

PRACTICE PROBLEMS:

1. The average weight of five packages is 30 pounds. Four of the packages weigh 10, 20, 40, and 45 pounds. What is the weight of the fifth package (in pounds)?

(A) 15 (B) 25 (C) 35 (D) 39 (E) 48

2. A is a number. The numbers $A-2$, $A-1$, $A+1$, and $A+2$ are averaged. What is the average (arithmetic mean) of these four numbers?

(A) $A+0.5$ (B) $A-0.5$ (C) A (D) 0 (E) 1

3. Carol needs to buy six gifts for Christmas. She has \$100 to spend on these gifts. She has already bought four gifts, costing \$15, \$20, \$12, and \$13. What is the average amount that she can spend on the remaining gifts?

(A) \$40 (B) \$20 (C) \$15 (D) \$17 (E) \$30

4. A survey of Town X found an average (arithmetic mean) of 3.2 persons per household and a mean of 1.2 televisions per household. If 48,000 people live in Town X, how many televisions are in Town X?

(A) 15,000 (B) 16,000 (C) 18,000 (D) 40,000 (E) 57,600

5. The average (arithmetic mean) of the first 3 numbers in a certain series is 10. If the average of the first 2 numbers is 10, what is the third number?

(A) 0(B) 5 (C) 10 (D) 20 (E) It cannot be determined from the information given.

6. If x is the average (arithmetic mean) of a and b, then in terms of x and c, what is the average of a, b, and c?

(A) $x+c$ (B) $\frac{x+c}{2}$ (C) $\frac{x+c}{3}$ (D) $\frac{2x+c}{2}$ (E) $\frac{2x+c}{3}$