

SECTION 7.7 Finding the Volume of a Solid of Revolution—Shells

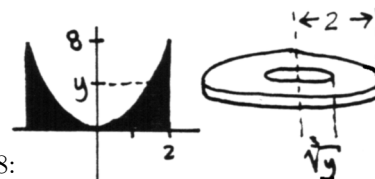
IN-SECTION EXERCISES:

EXERCISE 1.

Note that $y = x^3 \iff x = \sqrt[3]{y}$.

Choose a value of y between 0 and 8. A typical ‘disk with hole’ at this distance y has outer radius 2 and inner radius $\sqrt[3]{y}$. The ‘slice’ has thickness dy . The volume of the ‘slice’ is given by:

$$\pi(2^2)dy - \pi(\sqrt[3]{y})^2 dy = \pi(4 - y^{2/3})dy$$



The desired volume is found by ‘summing’ these slices as y travels from 0 to 8:

$$\int_0^8 \pi(4 - y^{2/3}) dy = \pi(4y - \frac{3}{5}y^{5/3}) \Big|_0^8 = \pi(32 - \frac{3}{5}8^{5/3}) = \pi(32 - \frac{3}{5}(8^{1/3})^5) = \pi(32 - \frac{3}{5}(2^5)) = \frac{64\pi}{5}$$

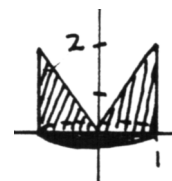
EXERCISE 2.

$$\begin{aligned} \text{desired volume} &= 2 \int_0^r 2\pi x \sqrt{r^2 - x^2} dx && \text{(shell formula)} \\ &= 4\pi \int_0^r x \sqrt{r^2 - x^2} dx && \text{(pull out constant } 2\pi) \\ &= \frac{4\pi}{(-2)} \int_0^r (-2)x \sqrt{r^2 - x^2} dx && \text{(multiply by 1 in form } \frac{-2}{-2}, \text{ linearity)} \\ &= -2\pi \int_{r^2}^0 u^{1/2} du && \text{(rename in terms of } u, \text{ new limits)} \\ &= -2\pi \cdot \frac{2}{3} u^{3/2} \Big|_{r^2}^0 && \text{(Simple Power Rule)} \\ &= -\frac{4\pi}{3} [0 - (r^2)^{3/2}] && \text{(evaluate antiderivative)} \\ &= -\frac{4\pi}{3} (-r^3) && \text{(simplify)} \\ &= \frac{4}{3}\pi r^3 && \text{(simplify)} \end{aligned}$$

END-OF-SECTION EXERCISES:

- Using shells:

$$\int_0^1 2\pi x(2x) dx = 4\pi \frac{x^3}{3} \Big|_0^1 = \frac{4\pi}{3}(1 - 0) = \frac{4\pi}{3}$$

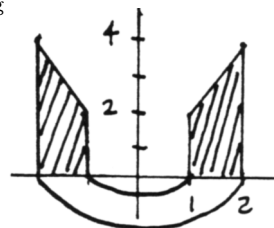


Using horizontal disks: $y = 2x \iff x = \frac{y}{2}$.

$$\int_0^2 \pi(1^2 - (\frac{y}{2})^2) dy = \int_0^2 \pi(1 - \frac{y^2}{4}) dy = \pi(y - \frac{1}{4} \frac{y^3}{3}) \Big|_0^2 = \pi(2 - \frac{2}{3}) = \frac{4\pi}{3}$$

2. Using shells:

$$\int_1^2 2\pi x(2x) dx = 4\pi \frac{x^3}{3} \Big|_1^2 = \frac{4\pi}{3}(2^3 - 1^3) = \frac{4\pi}{3}(7) = \frac{28\pi}{3}$$



Using disks: The bottom section has volume:

$$\pi(2^2 - 1^2)2 = 2\pi(3) = 6\pi$$

For the top section, choose a value of y between 2 and 4. The 'slice' at this value of y has outer radius 2 and inner radius $\frac{y}{2}$. Summing these 'slices' yields:

$$\int_2^4 \pi(2^2 - (\frac{y}{2})^2) dy = \int_2^4 \pi(4 - \frac{y^2}{4}) dy = \pi(4y - \frac{1}{4} \frac{y^3}{3}) \Big|_2^4 = \pi[(16 - \frac{16}{3}) - (8 - \frac{2}{3})] = \frac{10\pi}{3}$$

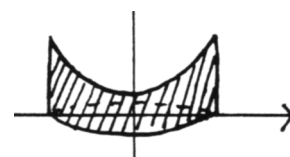
Thus, the total volume is:

$$6\pi + \frac{10\pi}{3} = \frac{28\pi}{3}$$

Which was easier?

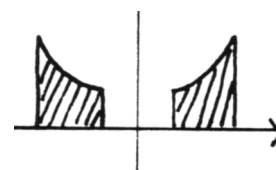
3. Using shells:

$$\begin{aligned} \int_0^1 2\pi x e^x dx &= 2\pi \int_0^1 x e^x dx = 2\pi [x e^x \Big|_0^1 - \int_0^1 e^x dx] \\ &= 2\pi [e - (e^x \Big|_0^1)] = 2\pi [e - (e^1 - e^0)] \\ &= 2\pi \end{aligned}$$



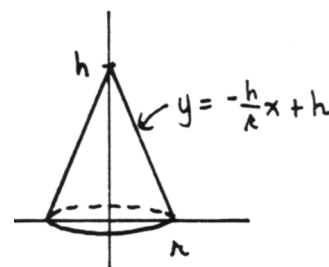
4. Using shells:

$$\begin{aligned} \int_1^2 2\pi x e^x dx &= 2\pi [x e^x \Big|_1^2 - \int_1^2 e^x dx] \\ &= 2\pi [(2e^2 - e) - (e^x \Big|_1^2)] \\ &= 2\pi [2e^2 - e - (e^2 - e)] \\ &= 2\pi(e^2) \end{aligned}$$



5. Generate the right circular cone by revolving
- $y = -\frac{h}{r}x + h$
- about the
- y
- axis. Using shells, the volume is:

$$\begin{aligned} \int_0^r 2\pi x (-\frac{h}{r}x + h) dx &= -\frac{2\pi h}{r} \int_0^r x^2 dx + 2\pi h \int_0^r x dx \\ &= -\frac{2\pi h}{r} \frac{x^3}{3} \Big|_0^r + 2\pi h \cdot \frac{x^2}{2} \Big|_0^r \\ &= -\frac{2\pi h}{3r} (r^3) + \pi h (r^2) \\ &= \frac{1}{3} \pi h r^2 \end{aligned}$$



6. A typical 'shell' has height
- h
- . The volume is:

$$\int_0^r 2\pi x h dx = 2\pi h \frac{x^2}{2} \Big|_0^r = \pi h (r^2 - 0) = \pi r^2 h$$

