SECTION 7.7 Finding the Volume of a Solid of Revolution—Shells

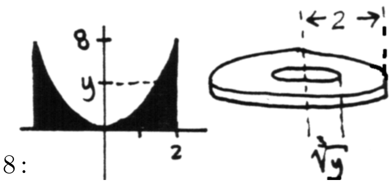
IN-SECTION EXERCISES:

EXERCISE 1.

Note that: $y = x^3 \iff x = \sqrt[3]{y}$

Choose a value of y between 0 and 8. A typical 'disk with hole' at this distance y has outer radius 2 and inner radius $\sqrt[3]{y}$. The 'slice' has thickness dy. The volume of the 'slice' is given by:

$$\pi(2^2)dy - \pi(\sqrt[3]{y})^2 dy = \pi(4 - y^{2/3})dy$$



The desired volume is found by 'summing' these slices as y travels from 0 to 8:

$$\int_0^8 \pi(4 - y^{2/3})\,dy = \pi(4y - \frac{3}{5}y^{5/3})\,\Big|_0^8 = \pi(32 - \frac{3}{5}8^{5/3}) = \pi(32 - \frac{3}{5}(8^{1/3})^5) = \pi(32 - \frac{3}{5}(2^5)) = \frac{64\pi}{5}$$

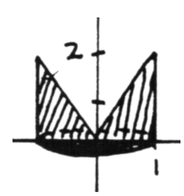
EXERCISE 2.

$$\begin{aligned}
\text{desired volume} &= 2\int_0^r 2\pi x\sqrt{r^2 - x^2}\,dx && \text{(shell formula)} \\
&= 4\pi \int_0^r x\sqrt{r^2 - x^2}\,dx && \text{(pull out constant } 2\pi\text{)} \\
&= \frac{4\pi}{(-2)}\int_0^r (-2)x\sqrt{r^2 - x^2}\,dx && \text{(multiply by 1 in form } \tfrac{-2}{-2}\text{, linearity)} \\
&= -2\pi \int_{r^2}^0 u^{1/2}\,du && \text{(rename in terms of } u\text{, new limits)} \\
&= -2\pi \cdot \frac{2}{3}u^{3/2}\Big|_{r^2}^0 && \text{(Simple Power Rule)} \\
&= -\frac{4\pi}{3}[0 - (r^2)^{3/2}] && \text{(evaluate antiderivative)} \\
&= -\frac{4\pi}{3}(-r^3) && \text{(simplify)} \\
&= \frac{4}{3}\pi r^3 && \text{(simplify)}
\end{aligned}$$

END-OF-SECTION EXERCISES:

1. Using shells:

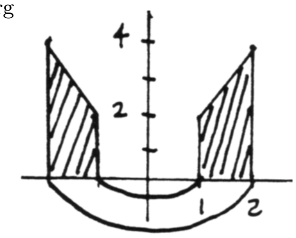
$$\int_0^1 2\pi x(2x)\,dx = 4\pi\frac{x^3}{3}\,\Big|_0^1 = \frac{4\pi}{3}(1 - 0) = \frac{4\pi}{3}$$



Using horizontal disks: $y = 2x \iff x = \frac{y}{2}$

$$\int_0^2 \pi(1^2 - (\frac{y}{2})^2)\,dy = \int_0^2 \pi(1 - \frac{y^2}{4})\,dy = \pi(y - \frac{1}{4}\frac{y^3}{3})\,\Big|_0^2 = \pi(2 - \frac{2}{3}) = \frac{4\pi}{3}$$

2. Using shells:



$$\int_1^2 2\pi x(2x)\,dx = 4\pi\frac{x^3}{3}\Big|_1^2 = \frac{4\pi}{3}(2^3-1^3) = \frac{4\pi}{3}(7) = \frac{28\pi}{3}$$

Using disks: The bottom section has volume:

$$\pi(2^2-1^2)2 = 2\pi(3) = 6\pi$$

For the top section, choose a value of y between 2 and 4. The 'slice' at this value of y has outer radius 2 and inner radius $\frac{y}{2}$. Summing these 'slices' yields:

$$\int_2^4 \pi(2^2-(\frac{y}{2})^2)\,dy = \int_2^4 \pi(4-\frac{y^2}{4})\,dy = \pi(4y-\frac{1}{4}\frac{y^3}{3})\Big|_2^4 = \pi[(16-\frac{16}{3})-(8-\frac{2}{3})] = \frac{10\pi}{3}$$

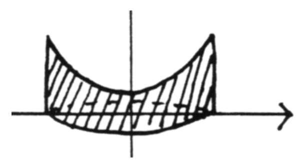
Thus, the total volume is:

$$6\pi+\frac{10\pi}{3} = \frac{28\pi}{3}$$

Which was was easier?

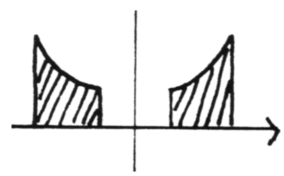
3. Using shells:

$$\begin{aligned}\int_0^1 2\pi x e^x\,dx &= 2\pi\int_0^1 xe^x\,dx = 2\pi[xe^x\Big|_0^1 - \int_0^1 e^x\,dx] \\ &= 2\pi[e-(e^x\Big|_0^1)] = 2\pi[e-(e^1-e^0)] \\ &= 2\pi\end{aligned}$$



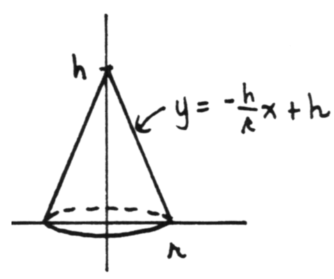
4. Using shells:

$$\begin{aligned}\int_1^2 2\pi x e^x\,dx &= 2\pi[xe^x\Big|_1^2 - \int_1^2 e^x\,dx] \\ &= 2\pi[(2e^2-e)-(e^x\Big|_1^2)] \\ &= 2\pi[2e^2-e-(e^2-e)] \\ &= 2\pi(e^2)\end{aligned}$$



5. Generate the right circular cone by revolving $y=-\frac{h}{r}x+h$ about the y-axis. Using shells, the volume is:

$$\begin{aligned}\int_0^r 2\pi x(-\frac{h}{r}x+h)\,dx &= -\frac{2\pi h}{r}\int_0^r x^2\,dx + 2\pi h\int_0^r x\,dx \\ &= -\frac{2\pi h}{r}\frac{x^3}{3}\Big|_0^r + 2\pi h\cdot\frac{x^2}{2}\Big|_0^r \\ &= -\frac{2\pi h}{3r}(r^3)+\pi h(r^2) \\ &= \frac{1}{3}\pi h r^2\end{aligned}$$



6. A typical 'shell' has height h. The volume is:

$$\int_0^r 2\pi x h\,dx = 2\pi h\frac{x^2}{2}\Big|_0^r = \pi h(r^2-0) = \pi r^2 h$$