SECTION 7.3 The Definite Integral as the Limit of Riemann Sums

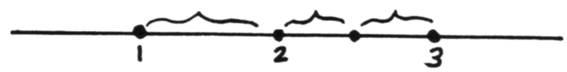
IN-SECTION EXERCISES:

EXERCISE 1.

motivation for the notation $\int_a^b f(x)\,dx$;
provides intuition to develop useful formulas involving the definite integral;
provides justification for numerical methods used to approximate the definite integral

EXERCISE 2.

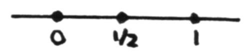
1. There are 4 points in the partition; the interval is divided into 3 subintervals.



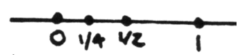
2. There are $n+1$ points, which divide the interval into n subintervals.

EXERCISE 3.

1. $P = \{0, \frac{1}{2}, 1\}$ is a partition of $[0,1]$ that has norm $\frac{1}{2}$. There are 3 points in this partition.



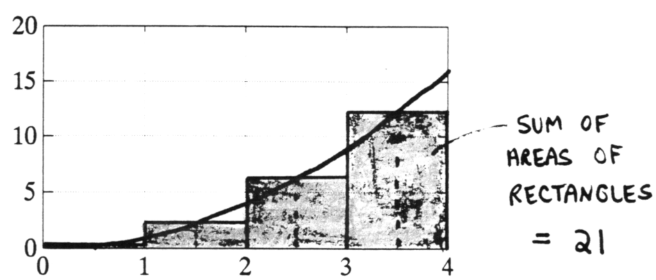
2. $P = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$ is another partition of $[0,1]$ that has norm $\frac{1}{2}$. This partition has 4 points.



3. Any partition of $[0,1]$ with norm $\frac{1}{2}$ must have at least 3 points. That is, 3 is the *fewest* number of points possible.

EXERCISE 4.

1.



2. $f(x_1^*) = f(.5) = (.5)^2 = 0.25$
$f(x_2^*) = f(1.5) = (1.5)^2 = 2.25$
$f(x_3^*) = f(2.5) = (2.5)^2 = 6.25$
$f(x_4^*) = f(3.5) = (3.5)^2 = 12.25$

3. Each rectangle has a width of 1 unit. Then:

$$R(P) = (1)[.25 + 2.25 + 6.25 + 12.25] = 21$$

4. The *actual* area is:

$$\int_0^4 x^2\,dx = \frac{x^3}{3}\Big|_0^4 = \frac{1}{3}(4^3) = \frac{64}{3} = 21\frac{1}{3}$$

The Riemann sum, in this case, gives a slight under-approximation to the definite integral.

EXERCISE 5.

The midpoints of the subintervals and their corresponding function values are summarized in the table below.

x_i^*	$f(x_i^*)$
0.2500	0.0625
0.7500	0.5625
1.2500	1.5625
1.7500	3.0625
2.2500	5.0625
2.7500	7.5625
3.2500	10.5625
3.7500	14.0625

Each rectangle has a width of $\frac{1}{2}$ units. Then:

$$R(P) = (\frac{1}{2})[.0625 + .5625 + 1.5625 + 3.0625 + 5.0625 + 7.5625 + 10.5625 + 14.0625] = \frac{1}{2}(42.5) = 21.25$$

Again, the Riemann sum gives a (very slight) under-approximation to the definite integral.

END-OF-SECTION EXERCISES:

1. EXP (an infinite class of functions)

2. EXP (a number)

3. SENTENCE; since $\int_0^1 x^2\,dx = \frac{1}{3}x^3\,\big|_0^1 = \frac{1}{3}(1-0) = \frac{1}{3}$, the sentence is true.

4. SENTENCE; conditional. If $f(x) \ge 0$ on $[a,b]$, then $\int_a^b f(x)\,dx$ gives the area between the graph of f and the x-axis on $[a,b]$. However, if $f(x) \le 0$ on $[a,b]$, then $\int_a^b f(x)\,dx$ gives *negative* the described area. And if f takes on both positive and negative values on $[a,b]$, it may be difficult to interpret the number $\int_a^b f(x)\,dx$ in terms of area.

5. SENTENCE; since e^x is always positive, this is TRUE.

6. SENTENCE; TRUE. (This is a consequence of the *definition* of the definite integral!)

7. SENTENCE; TRUE. The phrase 'g is twice differentiable' means that both g' and g'' exist. In particular, since g' is differentiable, g' must also be continuous. Thus, the integral $\int_a^b g'(x)\,dx$ is defined. To evaluate it, find a function that has derivative g'; of course, g is such a function! Then, by the Fundamental Theorem of Integral Calculus, $\int_a^b g'(x)\,dx = g(x)\big|_a^b = g(b) - g(a)$.

8. SENTENCE; TRUE. The function $|f(x)|$ is nonnegative. Thus, its graph lies on or above the x-axis. Therefore, the number $\int_a^b |f(x)|\,dx$ gives the area beneath the graph of $|f(x)|$, which is a nonnegative number.

9. SENTENCE; TRUE. The function $-|f(x)|$ is nonpositive. Thus, its graph lies on or below the x-axis. Therefore, the number $\int_a^b (-|f(x)|)\,dx$ gives *negative* the area trapped between the graph of $y = -|f(x)|$ and the x-axis, which is a nonpositive number.

10. SENTENCE; TRUE. Only the dummy variable has changed.