SECTION 7.2 The Definite Integral

IN-SECTION EXERCISES:

EXERCISE 1.

1. FALSE. The definite integral is a NUMBER.
2. TRUE.
3. FALSE. The indefinite integral is a class of functions; the definite integral is a NUMBER.
4. TRUE. The actual definition is presented in section 7.3.

EXERCISE 2.

1. $$\int_0^1 x^5\,dx = \frac{1}{6}x^6\,\Big|_0^1 = \frac{1}{6}(1^6 - 0^6) = \frac{1}{6}$$

   Since x^5 is nonnegative on the interval $[0,1]$, the number $\frac{1}{6}$ gives the area under the graph of $y = x^5$ on $[0,1]$.

2. $$\int_0^4 e^x\,dx = e^x\,\Big|_0^4 = e^4 - e^0 = e^4 - 1 \approx 53.6$$

   Since e^x is positive (everywhere), the number $e^4 - 1$ gives the area under the graph of $y = e^x$ on $[0,4]$.

3. $$\int_{-4}^0 e^x\,dx = e^x\big|_{-4}^0 = e^0 - e^{-4} = 1 - \frac{1}{e^4} \approx 0.98$$

   Since e^x is positive (everywhere), the number $1 - \frac{1}{e^4}$ gives the area under the graph of $y = e^x$ on $[-4,0]$.

4. $$\int_1^2 \frac{1}{x}\,dx = \ln|x|\,\Big|_1^2 = \ln 2 - \ln 1 = \ln 2 \approx 0.69$$

   Since $\frac{1}{x}$ is positive on $[1,2]$, the number $\ln 2$ gives the area under the graph of $y = \frac{1}{x}$ on $[1,2]$.

5. $$\int_{1/2}^2 \frac{1}{x}\,dx = \ln|x|\,\Big|_{1/2}^2 = \ln 2 - \ln(1/2) = \ln 2 - \ln 2^{-1} = \ln 2 + \ln 2 = 2\ln 2 \approx 1.39$$

   Since $\frac{1}{x}$ is positive on $[\frac{1}{2},2]$, the number $2\ln 2$ gives the area under the graph of $y = \frac{1}{x}$ on $[\frac{1}{2},2]$.

EXERCISE 3.

$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_1^b f(s)\,ds$; only different dummy variables have been used.

However, the integral $\int_a^b f(x)\,dx$ gives no information about $\int_c^d f(x)\,dx$; unless $a = c$ and $b = d$. Here, the function f is being integrated over a *different* interval.

The integral $\int_a^b f(x)\,dx$ gives no information about $\int_a^b g(x)\,dx$, unless $f = g$ on $[a,b]$. A different function is being integrated.

EXERCISE 4.

1. $\int_{-3}^{-2} f(x)\,dx = A$
2. $\int_{-3}^{0} f(t)\,dt = A - 2A = -A$. The area beneath the x-axis is treated as negative by the definite integral.
3. $\int_{-3}^{2} f(s)\,ds = A - 2A + 3A = 2A$
4. This integral cannot be computed exactly with only the given information; however, one would suspect that $\int_0^5 f(x)\,dx$ is a negative number.
5. $\int_{-2}^{2} f(t)\,dt = -2A + 3A = A$
6. This integral cannot be computed exactly with only the given information; however, one would suspect that $\int_{-3}^{-1} f(y)\,dy \approx 0$.

EXERCISE 5.

Method I: First solve the companion indefinite integral problem:

$u = 5x + 1$
$du = 5dx$

$$\int \frac{1}{5x+1}\,dx = \frac{1}{5}\int \frac{1}{u}\,du = \frac{1}{5}\ln|5x+1| + C\ ;$$

then use the simplest antiderivative to evaluate the definite integral:

$$\int_0^1 \frac{1}{5x+1}\,dx = \frac{1}{5}\ln|5x+1|\,\Big|_0^1 = \frac{1}{5}(\ln 6 - \ln 1) = \frac{1}{5}\ln 6$$

Method II: Solve the definite integral directly:

$$\begin{aligned}\int_0^1 \frac{1}{5x+1}\,dx &= \int_0^1 \frac{1}{5(x+\frac{1}{5})}\,dx = \frac{1}{5}\int_0^1 \frac{1}{x+\frac{1}{5}}\,dx \\ &= \frac{1}{5}\ln|x+\frac{1}{5}|\,\Big|_0^1 = \frac{1}{5}(\ln\frac{6}{5} - \ln\frac{1}{5}) \\ &= \frac{1}{5}(\ln 6 - \ln 5 + \ln 5) = \frac{1}{5}\ln 6\end{aligned}$$

Of course, the answers agree!

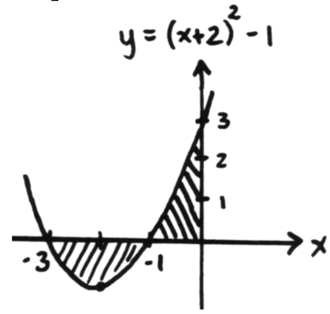
EXERCISE 6.

$$\int_0^3 (-(x-2)^2+1)\,dx = -\frac{1}{3}(x-2)^3 + x\,\Big|_0^3 = [-\frac{1}{3}+3] - [-\frac{1}{3}(-8)] = \frac{8}{3} - \frac{8}{3} = 0$$

The integral is 0, because the magnitude of area above the x-axis is the same as the magnitude of the area below the x-axis, on the interval $[0,3]$.

EXERCISE 7.
The desired area must be found in two pieces:

$$\begin{aligned}\int_{-3}^{-1} [(x+2)^2-1]\,dx &= [\frac{(x+2)^3}{3} - x]\,\Big|_{-3}^{-1} \\ &= \left[\frac{(-1+2)^3}{3} - (-1)\right] - \left[\frac{(-3+2)^3}{3} - (-3)\right] \\ &= (\frac{1}{3}+1) - (-\frac{1}{3}+3) = \frac{4}{3} - \frac{8}{3} = -\frac{4}{3}\end{aligned}$$



The answer is negative, since the area is beneath the x-axis.

$$\begin{aligned}\int_{-1}^{0} [(x+2)^2-1]\,dx &= [\frac{(x+2)^3}{3} - x]\,\Big|_{-1}^{0} \\ &= \frac{2^3}{3} - (\frac{1}{3} - (-1)) \\ &= \frac{8}{3} - \frac{4}{3} = \frac{4}{3}\end{aligned}$$

The desired area is $\frac{4}{3} + \frac{4}{3} = \frac{8}{3}$.

END-OF-SECTION EXERCISES:

1. $$\int_0^2 \frac{3}{2}x^4\,dx = \frac{3}{2}\frac{x^5}{5}\,\Big|_0^2 = \frac{3}{10}(2^5 - 0^5) = \frac{48}{5}$$

2. $$\int_1^8 t^{1/3}\,dt = \frac{3}{4}t^{4/3}\,\Big|_1^8 = \frac{3}{4}(8^{4/3} - 1^{4/3}) = \frac{3}{4}((8^{1/3})^4 - 1) = \frac{3}{4}(16-1) = \frac{45}{4}$$

3. $$\int_{-1}^1 (2x-3)\,dx = (2\frac{x^2}{2} - 3x)\,\Big|_{-1}^1 = (1-3) - (1+3) = -2 - 4 = -6$$

4.
$$\int_0^1 (ax+b)\,dx = (a\frac{x^2}{2} + bx)\,\Big|_0^1 = \frac{a}{2} + b$$

5. First, find the companion indefinite integral:

$$\int \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\int \frac{1}{1+x^3}(3x^2)\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|1+x^3| + C\ ;$$

now use the simplest antiderivative to evaluate the definite integral:

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\ln|1+x^3|\,\Big|_0^1 = \frac{1}{3}(\ln 2 - \ln 1) = \frac{1}{3}\ln 2$$

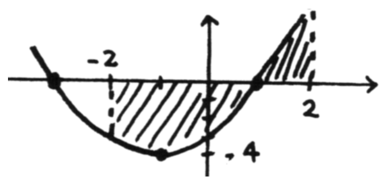
6.
$$\begin{aligned}\int_{\ln 2}^{\ln 3} e^{2t}\,dt &= \frac{1}{2}e^{2t}\,\Big|_{\ln 2}^{\ln 3} \\ &= \frac{1}{2}(e^{2\ln 3} - e^{2\ln 2}) = \frac{1}{2}(e^{\ln 3^2} - e^{\ln 2^2}) \\ &= \frac{1}{2}(9-4) = \frac{5}{2}\end{aligned}$$

7.
$$\int_0^2 (1+e^x)\,dx = (x+e^x)\,\Big|_0^2 = (2+e^2) - (0+e^0) = 1+e^2$$

8. The graph of f is a parabola that crosses the x-axis at $x=1$ and $x=-3$. Using calculus to find the vertex: $f(x) = x^2+2x-3$, so $f'(x) = 2x+2$;

$$f'(x) = 0 \quad\Longleftrightarrow\quad 2x+2=0 \quad\Longleftrightarrow\quad x=-1$$

Also, $f(-1) = (-1-1)(-1+3) = (-2)(2) = -4$. Since $f''(x) = 2$, the graph is always concave up.



The desired area must be found in two pieces:

$$\begin{aligned}\int_{-2}^1 f(x) = \int_{-2}^1 (x^2+2x-3)\,dx &= (\frac{x^3}{3} + x^2 - 3x)\,\Big|_{-2}^1 \\ &= (\frac{1}{3} + 1 - 3) - (-\frac{8}{3} + 4 + 6) \\ &= -\frac{5}{3} - \frac{22}{3} = -\frac{27}{3} = -9\end{aligned}$$

The answer is negative, because the area lies beneath the x-axis. Then:

$$\begin{aligned}\int_1^2 (x^2+2x-3)\,dx &= (\frac{x^3}{3} + x^2 - 3x)\,\Big|_1^2 = (\frac{8}{3} + 4 - 6) - (\frac{1}{3} + 1 - 3) \\ &= \frac{2}{3} - (-\frac{5}{3}) = \frac{7}{3}\end{aligned}$$

The desired area is $9 + \frac{7}{3} = 11\frac{1}{3}$.

9. First, factor f: find A and B with $AB = (2)(-3) = -6$ and $A+B = 5$; take $A = 6$ and $B = -1$. Then:

$$2x^2 + 5x - 3 = 2x^2 + 6x - x - 3 = 2x(x+3) - (x+3) = (2x-1)(x+3)$$

The graph of f is a parabola that crosses the x-axis at $x = -3$ and $x = \frac{1}{2}$.
$f'(x) = 4x + 5$;
$f'(x) = 0 \iff 4x + 5 = 0 \iff x = -\frac{5}{4}$.
$f(-\frac{5}{4}) = 2(-\frac{5}{4})^2 + 5(-\frac{5}{4}) - 3 = \ldots = -\frac{49}{8}$.
Since $f''(x) = 4 > 0$, the graph is concave up everywhere.
The desired area must be found in two pieces:

$$\begin{aligned}\int_0^{1/2} (2x^2 + 5x - 3)\,dx &= (2\frac{x^3}{3} + 5\frac{x^2}{2} - 3x)\,\Big|_0^{1/2} \\ &= [\frac{2}{3}(\frac{1}{2})^3 + \frac{5}{2}(\frac{1}{2})^2 - 3(\frac{1}{2})] - 0 \\ &= \frac{1}{12} + \frac{5}{8} - \frac{3}{2} = \frac{2}{24} + \frac{15}{24} - \frac{36}{24} = -\frac{19}{24}\end{aligned}$$

The answer is negative, since the area lies beneath the x-axis. Then:

$$\begin{aligned}\int_{1/2}^{2} (2x^2 + 5x - 3)\,dx &= (2\frac{x^3}{3} + 5\frac{x^2}{2} - 3x)\,\Big|_{1/2}^{2} \\ &= [\frac{2}{3}(2^3) + \frac{5}{2}(2^2) - 3(2)] - (\frac{2}{3}(\frac{1}{2})^3 + \frac{5}{2}(\frac{1}{2})^2 - 3(\frac{1}{2})] \\ &= \cdots = \frac{81}{8}\end{aligned}$$

The desired area is $\frac{19}{24} + \frac{81}{8} = \frac{131}{12}$.