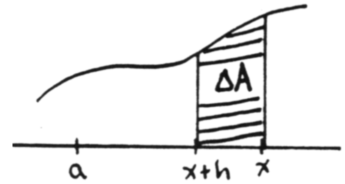
SECTION 7.1 Using Antiderivatives to find Area

IN-SECTION EXERCISES:

EXERCISE 1.



1. If h is a small negative number, then $x+h$ is a little to the left of x.
2. In this case, $\Delta A = A(x) - A(x+h)$.

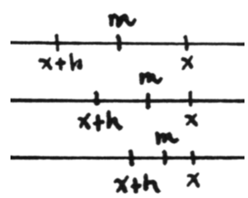
EXERCISE 2.

1. When h is negative, $-h$ is positive. In this case, the positive number $-h$ gives the width of the approximating rectangle.
2. The over-approximating rectangle has height $f(M)$ and width $-h$, hence area $f(M)\cdot(-h)$.
3.

$$\begin{aligned}
f(m)(-h) \le \Delta A \le f(M)(-h) \quad &\iff \quad f(m) \le \frac{\Delta A}{-h} \le f(M) && \text{(divide by } -h > 0\text{)} \\
&\iff \quad f(m) \le \frac{A(x)-A(x+h)}{-h} \le f(M) && \text{(definition of } \Delta A\text{)} \\
&\iff \quad f(m) \le \frac{A(x+h)-A(x)}{h} \le f(M) && \text{(multiply quotient by } \frac{(-1)}{(-1)}\text{)}
\end{aligned}$$

EXERCISE 3.

Now let h approach 0 (from the left-hand side, since h is negative). Remember that m is trapped in the interval $[x+h, x]$, so as h approaches zero, m is forced to get close to x. That is, as $h \to 0^-$, it must be that $m \to x^-$.



EXERCISE 4.

By hypothesis, f is continuous at x. Therefore, when the inputs are close to x, the corresponding outputs must be close to $f(x)$. In particular, when m is close to x, $f(m)$ must be close to $f(x)$. More precisely, as $m \to x^-$, we must have $f(m) \to f(x)$.

Similarly, since M is trapped between $x+h$ and x, as h approaches 0, M must approach x. And as M gets close to x, the continuity of f at x tells us that $f(M)$ approaches $f(x)$.

Reconsider the previous inequality in light of our new information:

$$f(m) \le \frac{A(x+h)-A(x)}{h} \le f(M)$$

As h approaches 0 (from the left-hand side), both $f(m)$ and $f(M)$ are approaching $f(x)$. So the quotient

$$\frac{A(x+h)-A(x)}{h}$$

is pinched between numbers which are *both* going to the *same number*, $f(x)$! Therefore, $\frac{A(x+h)-A(x)}{h}$ must also be getting close to $f(x)$! That is, it must be that

$$\lim_{h\to 0^-} \frac{A(x+h)-A(x)}{h} = f(x)\ .$$

EXERCISE 5.

1. It need only be shown that F is a function which, when differentiated, yields $2x$: $F'(x) = 2x$.
2. Now, $F(3) - F(0) = (3^2 + 7) - (0^2 + 7) = 9 + 7 - 0 - 7 = 9$. The '7' cancels out in the evaluation process.

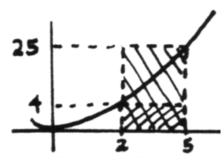
EXERCISE 6.

1. area of trapezoid $= \frac{1}{2}(4-1)(2+8) = \frac{1}{2}(3)(10) = 15$.
2. An antiderivative of $f(x) = 2x$ is $F(x) = x^2$. Then, $F(4) - F(1) = 4^2 - 1^2 = 16 - 1 = 15$. Compare answers!

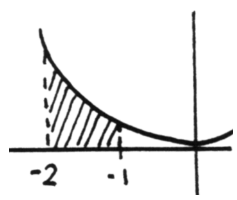
EXERCISE 7.

1. under-approximation: $(5-2)(4) = 12$

   over-approximation: $(3)(25) = 75$
2. An antiderivative of $f(x) = x^2$ is $F(x) = \frac{x^3}{3}$. Then, $F(5) - F(2) = \frac{5^3}{3} - \frac{2^3}{3} = \frac{117}{3} = 39$. Certainly believable, based on the earlier estimate!
3. Using $F(x) = \frac{x^3}{3} + 1$, $F(5) - F(2) = (\frac{5^3}{3} + 1) - (\frac{2^3}{3} + 1) = 39$.

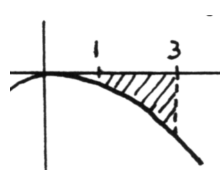


EXERCISE 8.
Take $F(x) = \frac{x^3}{3}$. Then, $F(-1) - F(-2) = \frac{(-1)^3}{3} - \frac{(-2)^3}{3} = -\frac{1}{3} - (-\frac{8}{3}) = -\frac{1}{3} + \frac{8}{3} = \frac{7}{3}$.



EXERCISE 9.

1.



2. Take $F(x) = -\frac{x^3}{3}$. Then, $F(3) - F(1) = (-\frac{3^3}{3}) - (-\frac{1^3}{3}) = -9 + \frac{1}{3} = -8\frac{2}{3}$.

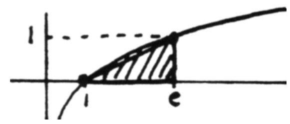
   The area under the graph of $f(x) = x^2$ on $[1, 3]$ is found by using the antiderivative $G(x) = \frac{x^3}{3}$:

   $G(3) - G(1) = \frac{3^3}{3} - \frac{1^3}{3} = 9 - \frac{1}{3} = 8\frac{2}{3}$.

   Note that the two answers differ only by a sign. In one case, the area is above the x-axis; in the other case, the area has the same magnitude, but is below the x-axis.
3. Conjecture: the definite integral treats area below the x-axis as negative.

END-OF-SECTION EXERCISES:

1.



approximation by a triangle: $\frac{1}{2}(1)(e-1) \approx 0.86$.

actual area: Using integration by parts, an antiderivative of $f(x) = \ln x$ is $F(x) = x \ln x - x$. Then:

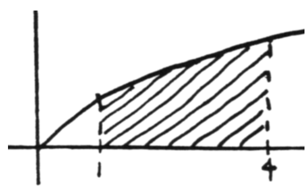
$$F(e) - F(1) = (e \ln e - e) - (1 \ln 1 - 1) = (e - e) - (0 - 1) = 1$$

2.

approximation by a trapezoid: $\frac{1}{2}(2-1)(1+\frac{1}{2}) = \frac{1}{2}(\frac{3}{2}) = \frac{3}{4}$.

actual area: An antiderivative of $f(x) = \frac{1}{x}$ is $F(x) = \ln|x|$. Then, $F(2) - F(1) = \ln 2 - \ln 1 = \ln 2 \approx 0.69$.

3.



approximation by a trapezoid: $\frac{1}{2}(4-1)(1+2) = \frac{1}{2}(9) = \frac{9}{2} = 4.5$.

actual area: An antiderivative of $f(x) = \sqrt{x} = x^{1/2}$ is $F(x) = \frac{2}{3}x^{3/2} = \frac{2}{3}\sqrt{x^3}$. Then, $F(4) - F(1) = \frac{2}{3}\sqrt{4^3} - \frac{2}{3}\sqrt{1^3} = \frac{2}{3}(8) - \frac{2}{3}(1) = \frac{2}{3}(7) = \frac{14}{3} \approx 4.67$.

4. approximation by a triangle: $\frac{1}{2}(1)(2-1) = \frac{1}{2} = 0.5$.

There are several correct approaches. Here, we'll find the area under $y = x^2 + 1$, and subtract off the area of the rectangle.

An antiderivative of $f(x) = x^2 + 1$ is $F(x) = \frac{x^3}{3} + x$. Then, $F(1) - F(0) = \frac{1}{3} + 1 - 0 = 1\frac{1}{3}$ is the area under the graph of f on $[0, 1]$. Subtracting off the area of the rectangle yields the desired result: $1\frac{1}{3} - (1)(1) = \frac{1}{3}$.