SECTION 6.6 Integration by Parts Formula

IN-SECTION EXERCISES:

EXERCISE 2.

$$\frac{d}{dx}(xe^x - e^x) = (xe^x + (1)e^x) - e^x = xe^x$$

EXERCISE 3.

1. The choice for dv must be something that we know how to integrate. Here are the possible choices, with the corresponding choices for u:

$$\begin{aligned} dv = dx \quad &\text{with corresponding} \quad u = xe^{-x} \\ dv = x\,dx \quad &\text{with corresponding} \quad u = e^{-x} \\ dv = e^{-x}\,dx \quad &\text{with corresponding} \quad u = x \end{aligned}$$

2. The only case where $\frac{du}{dx}$ is simpler than u is when $u = x$.
   This is how the choices $u = x$ and $dv = e^{-x}\,dx$ were arrived at.

EXERCISE 4.

1.

$$\begin{aligned} \int x \ln x\,dx &= (\ln x)(\frac{x^2}{2}) - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + C = \frac{1}{2}x^2(\ln x - \frac{1}{2}) + C \end{aligned}$$

$u = \ln x \quad dv = x\,dx$

$du = \frac{1}{x}dx \quad v = \frac{x^2}{2}$

2.

$$\begin{aligned} \int xe^{3x}\,dx &= x(\frac{1}{3}e^{3x}) - \int \frac{1}{3}e^{3x}\,dx \\ &= \frac{1}{3}xe^{3x} - \frac{1}{3} \cdot \frac{1}{3}e^{3x} + C \\ &= \frac{1}{9}e^{3x}(3x - 1) + C \end{aligned}$$

$u = x \quad dv = e^{3x}dx$

$du = dx \quad v = \frac{1}{3}e^{3x}$

3.

$$\begin{aligned} \int x^3 \ln x\,dx &= (\ln x)(\frac{x^4}{4}) - \int \frac{x^4}{4} \cdot \frac{1}{x}\,dx = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{4} \cdot \frac{x^4}{4} + C = \frac{1}{16}x^4(4\ln x - 1) + C \end{aligned}$$

$u = \ln x \quad dv = x^3dx$

$du = \frac{1}{x}dx \quad v = \frac{x^4}{4}$

4. $\displaystyle\int \ln 3x\,dx = (\ln 3x)(x) - \int x \cdot \frac{1}{x}\,dx = x \ln x - x + C$

$u = \ln 3x \quad dv = dx$

$du = \frac{1}{3x} \cdot 3\,dx \quad v = x$

$= \frac{1}{x}dx$

EXERCISE 5.

1.

$$\begin{aligned} \frac{d}{dx}[(x+3)\ln(x+3) - x] &= (x+3)\frac{1}{x+3}(1) + (1)\ln(x+3) - 1 \\ &= 1 + \ln(x+3) - 1 = \ln(x+3) \end{aligned}$$

2.

$$\int 3\ln(x+1)\,dx = 3\int \ln(x+1)\,dx$$

$u = \ln(x+1)$ $\quad dv = dx$

$du = \frac{1}{x+1}\,dx$ $\quad v = x+1$

$$\begin{aligned}
&= 3\left[(x+1)\ln(x+1) - \int (x+1)\frac{1}{x+1}\,dx\right] \\
&= 3\big[(x+1)\ln(x+1) - x + C\big] \\
&= 3\big[(x+1)\ln(x+1) - x\big] + K
\end{aligned}$$

3.

$u = \ln(t-\frac{1}{2})$ $\quad dv = dt$

$du = \frac{1}{t-\frac{1}{2}}\,dt$ $\quad v = t - \frac{1}{2}$

$$\begin{aligned}
\int \ln(t-\tfrac{1}{2})\,dt &= (t-\tfrac{1}{2})\ln(t-\tfrac{1}{2}) - \int (t-\tfrac{1}{2})\frac{1}{t-\frac{1}{2}}\,dt \\
&= (t-\tfrac{1}{2})\ln(t-\tfrac{1}{2}) - t + C
\end{aligned}$$

EXERCISE 6.

2.

$u = x^2$ $\quad dv = e^{3x}$

$du = 2x\,dx$ $\quad v = \frac{1}{3}e^{3x}$

$$\begin{aligned}
\int x^2e^{3x}\,dx &= x^2(\tfrac{1}{3}e^{3x}) - \int \tfrac{1}{3}e^{3x}(2x)\,dx \\
&= \tfrac{1}{3}x^2e^{3x} - \tfrac{2}{3}\int xe^{3x}\,dx\ ;
\end{aligned}$$

$u = x$ $\quad dv = e^{3x}$

$du = dx$ $\quad v = \frac{1}{3}e^{3x}$

$$\begin{aligned}
\int xe^{3x}\,dx &= (x)(\tfrac{1}{3}e^{3x}) - \int \tfrac{1}{3}e^{3x}\,dx \\
&= \tfrac{1}{3}xe^{3x} - \tfrac{1}{3}\cdot\tfrac{1}{3}e^{3x} + C \\
&= \tfrac{1}{9}e^{3x}(3x-1) + C
\end{aligned}$$

Combining results:

$$\begin{aligned}
\int x^2e^{3x}\,dx &= \tfrac{1}{3}x^2e^{3x} - \tfrac{2}{3}\big[\tfrac{1}{9}e^{3x}(3x-1)\big] + C \\
&= \tfrac{1}{27}e^{3x}[9x^2 - 2(3x-1)] + C \\
&= \tfrac{1}{27}e^{3x}[9x^2 - 6x + 2] + C
\end{aligned}$$

EXERCISE 7.

1.

$$\int \frac{x}{(1+x)^6}\,dx = \int \frac{u-1}{u^6}\,du = \int (u^{-5} - u^{-6})\,du$$

$u = 1 + x;\ x = u - 1$

$du = dx$

$$= \frac{u^{-4}}{-4} - \frac{u^{-5}}{-5} + C = -\frac{1}{4(1+x)^4} + \frac{1}{5(1+x)^5} + C$$

Checking:

$$\begin{aligned}
\frac{d}{dx}\Big(-\frac{1}{4(1+x)^4} + \frac{1}{5(1+x)^5}\Big) &= \frac{d}{dx}\Big(-\frac{1}{4}(1+x)^{-4} + \frac{1}{5}(1+x)^{-5}\Big) \\
&= -\frac{1}{4}\cdot(-4)(1+x)^{-5} + \frac{1}{5}\cdot(-5)(1+x)^{-6} \\
&= \frac{1}{(1+x)^5}\cdot\frac{(1+x)}{(1+x)} - \frac{1}{(1+x)^6} = \frac{x}{(1+x)^6}
\end{aligned}$$

2.

$$\int \frac{x}{(1+x)^6}\,dx = x\left(\frac{(1+x)^{-5}}{-5}\right) - \int \frac{(1+x)^{-5}}{-5}\,dx$$

$u = x \quad dv = \frac{1}{(1+x)^6}dx$

$du = dx \quad v = \frac{(1+x)^{-5}}{-5}$

$$\begin{aligned}
&= -\frac{x}{5(1+x)^5} + \frac{1}{5}\int (1+x)^{-5}\,dx \\
&= -\frac{x}{5(1+x)^5} + \frac{1}{5}\cdot\frac{(1+x)^{-4}}{-4} + C \\
&= -\frac{x}{5(1+x)^5} - \frac{1}{20(1+x)^4} + C
\end{aligned}$$

Differentiating:

$$\begin{aligned}
\frac{d}{dx}\left[-\frac{x}{5(1+x)^5} - \frac{1}{20(1+x)^4}\right] &= \frac{d}{dx}\left[-\frac{1}{5}x(1+x)^{-5} - \frac{1}{20}(1+x)^{-4}\right] \\
&= -\frac{1}{5}[x(-5)(1+x)^{-6} + (1)(1+x)^{-5}] - \frac{1}{20}(-4)(1+x)^{-5} \\
&= \frac{x}{(1+x)^6} - \frac{1}{5(1+x)^5} + \frac{1}{5(1+x)^5} \\
&= \frac{x}{(1+x)^6}
\end{aligned}$$

3. Rewriting the solution to (1) as a single fraction with denominator $20(1+x)^5$ yields:

$$-\frac{1}{4(1+x)^4}\cdot\frac{5(1+x)}{5(1+x)} + \frac{1}{5(1+x)^5}\cdot\frac{4}{4} = \frac{-5(1+x)+4}{20(1+x)^5} = \frac{-5x-1}{20(1+x)^5}$$

Doing the same with (2):

$$\begin{aligned}
-\frac{x}{5(1+x)^5}\cdot\frac{4}{4} - \frac{1}{20(1+x)^4}\cdot\frac{(1+x)}{(1+x)} &= \frac{-4x-1-x}{20(1+x)^5} \\
&= \frac{-5x-1}{20(1+x)^5}
\end{aligned}$$

Compare!

END-OF-SECTION EXERCISES:

1.

$$\begin{aligned}
\int (e^x-1)^2\,dx &= \int (e^{2x} - 2e^x + 1)\,dx \\
&= \frac{1}{2}e^{2x} - 2e^x + x + C
\end{aligned}$$

2.

$$\begin{aligned}
\int \frac{\ln(x^2+2x+1)}{x+1}\,dx &= \int \frac{\ln(x+1)^2}{x+1}\,dx \\
&= 2\int \frac{\ln(x+1)}{x+1}\,dx \\
&= 2\int u\,du = 2\frac{u^2}{2} + C = (\ln(x+1))^2 + C
\end{aligned}$$

$u = \ln(x+1)$

$du = \frac{1}{x+1}\,dx$

3.

$u = 1 + e^x$
$du = e^x\,dx$

$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{1}{u}\,du$$
$$= \ln|u| + C = \ln|1+e^x| + C$$

4.

$$\int \ln\frac{1+x}{x}\,dx = \int \ln(1+x) - \ln x\,dx$$
$$= [(x+1)\ln(x+1) - x] - [x\ln x - x] + C$$
$$= (x+1)\ln(x+1) - x\ln x + C$$

(PARTS, TWICE!)

5.

$$\int \sqrt{\frac{e^t}{2}}\,dt = \frac{1}{\sqrt{2}}\int (e^t)^{1/2}\,dt = \frac{1}{\sqrt{2}}\int e^{t/2}\,dt$$
$$= \frac{1}{\sqrt{2}}\cdot 2e^{t/2} + C = \sqrt{2}(e^t)^{1/2} + C$$
$$= \sqrt{2e^t} + C$$

6.

$$\int \frac{x}{\sqrt{x^4\ln x}}\,dx = \int \frac{x}{x^2\sqrt{\ln x}}\,dx = \int \frac{1}{x\sqrt{\ln x}}\,dx$$
$$= \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = 2u^{1/2} + C$$
$$= 2\sqrt{\ln x} + C$$

$u = \ln x$
$du = \frac{1}{x}\,dx$