SECTION 6.5 More on Substitution

IN-SECTION EXERCISES:

EXERCISE 1.

'Role Reversal':

$$\begin{aligned}\int \frac{3t}{t-1}\,dt &= 3\int \frac{t}{t-1}\,dt \\ &= 3\int \frac{u+1}{u}\,du = 3\int 1+\frac{1}{u}\,du \\ &= 3(u+\ln|u|)+C = 3u+3\ln|u|+C \\ &= 3(t-1)+3\ln|t-1|+C \\ &= 3t+3\ln|t-1|+K\end{aligned}$$

$u = t-1$; $t = u+1$

$du = dt$

Using long division:

$$\begin{aligned}\int \frac{3t}{t-1}\,dt &= 3\int \frac{t}{t-1}\,dt \\ &= 3\int 1+\frac{1}{t-1}\,dt = 3(t+\ln|t-1|)+C\end{aligned}$$

$$\begin{array}{r} 1 \\ t-1\,\overline{)\,t} \\ -(t-1) \\ \hline 1 \end{array}$$

EXERCISE 2.

'Role Reversal'

$$\begin{aligned}\int \frac{2t}{3t-1}\,dt &= 2\int \frac{t}{3t-1}\,dt = 2\int \frac{\frac{u+1}{3}}{u}\cdot\frac{du}{3} \\ &= \frac{2}{9}\int \frac{u+1}{u}\,du = \frac{2}{9}\int 1+\frac{1}{u}\,du \\ &= \frac{2}{9}(u+\ln|u|)+C \\ &= \frac{2}{9}(3t-1+\ln|3t-1|)+C \\ &= \frac{2}{9}(3t+\ln|3t-1|)+K\end{aligned}$$

$u = 3t-1$; $t = \frac{u+1}{3}$

$du = 3dt$; $dt = \frac{du}{3}$

Using long division:

$$\begin{aligned}\int \frac{2t}{3t-1}\,dt &= 2\int \frac{t}{3t-1}\,dt \\ &= 2\int \frac{1}{3}+\frac{1/3}{3t-1}\,dt \\ &= 2\int \frac{1}{3}\,dt+\frac{2}{3}\int \frac{1}{3t-1}\,dt \\ &= \frac{2}{3}t+\frac{2}{9}\int \frac{3}{3t-1}\,dt \\ &= \frac{2}{3}t+\frac{2}{9}\int \frac{1}{u}\,du \\ &= \frac{2}{3}t+\frac{2}{9}\ln|3t-1|+C\end{aligned}$$

$$\begin{array}{r} \frac{1}{3} \\ 3t-1\,\overline{)\,t} \\ -(t-\frac{1}{3}) \\ \hline \frac{1}{3} \end{array}$$

$u = 3t-1$

$du = 3\,dt$

Compare! As you get more proficient in integrating, you will probably be able to leave out a number of the steps included above.

EXERCISE 3.
Eventually, you should be above to solve these problems with the number of steps given here:

1.

$u = t+1$; $t = u-1$
$du = dt$

$$\int t(t+1)^7\,dt = \int (u-1)u^7\,du = \int u^8 - u^7\,du$$
$$= \frac{(t+1)^9}{9} - \frac{(t+1)^8}{8} + C$$

2.

$u = 3-2x$;
$x = \frac{u-3}{-2} = \frac{3-u}{2}$
$du = -2dx$;
$dx = \frac{du}{-2}$

$$\int \frac{5x}{\sqrt{(3-2x)^3}}\,dx = 5\int \frac{\frac{3-u}{2}}{u^{3/2}}\cdot\frac{du}{-2}$$
$$= -\frac{5}{4}\int (3-u)u^{-3/2}\,du$$
$$= -\frac{5}{4}\int 3u^{-3/2} - u^{-1/2}\,du$$
$$= -\frac{5}{4}(3\cdot\frac{u^{-1/2}}{-1/2} - \frac{u^{1/2}}{1/2}) + C$$
$$= -\frac{5}{4}(\frac{-6}{\sqrt{3-2x}} - 2\sqrt{3-2x}) + C$$

3.

$w = u^2+1$
$dw = 2u\,du$

$$\int u\sqrt{u^2+1}\,du = \frac{1}{2}\int \sqrt{w}\,dw$$
$$= \frac{1}{2}\cdot\frac{2}{3}w^{3/2} + C$$
$$= \frac{1}{3}\sqrt{(u^2+1)^3} + C$$

EXERCISE 4.

2.

$$\frac{d}{dx}(2\sqrt{x} - 2\ln|1+\sqrt{x}|) = 2\cdot\frac{1}{2}x^{-1/2} - 2\cdot\frac{1}{1+\sqrt{x}}\cdot\frac{1}{2}x^{-1/2}$$
$$= \frac{1}{\sqrt{x}}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}} - \frac{1}{\sqrt{x}(1+\sqrt{x})}$$
$$= \frac{1+\sqrt{x}-1}{\sqrt{x}(1+\sqrt{x})}$$
$$= \frac{1}{1+\sqrt{x}}$$

EXERCISE 5.
'Role Reversal':

$u = x-1$; $x = u+1$
$du = dx$

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u+1}{\sqrt{u}}\,du$$
$$= \int u^{1/2} + u^{-1/2}\,du$$
$$= \frac{2}{3}u^{3/2} + 2u^{1/2} + C$$
$$= \frac{2}{3}\sqrt{(x-1)^3} + 2\sqrt{x-1} + C$$

Rationalizing Substitution:

$u = \sqrt{x-1}$;
$u^2 = x-1$; $x = u^2+1$
$2u\frac{du}{dx} = 1$; $2u\,du = dx$

$$\begin{aligned}\int \frac{x}{\sqrt{x-1}}\,dx &= \int \frac{u^2+1}{u}\cdot 2u\,du\\ &= 2\int u^2+1\,du = 2(\frac{u^3}{3}+u)+C\\ &= \frac{2}{3}(\sqrt{x-1})^3 + 2\sqrt{x-1}+C\end{aligned}$$

Compare!

END-OF-SECTION EXERCISES:

1.
$$\int \frac{e^{2x}+1}{5}\,dx = \frac{1}{5}\int e^{2x}+1\,dx = \frac{1}{5}(\frac{1}{2}e^{2x}+x)+C$$

2.
$u = 3x^2-1$
$du = 6x\,dx$

$$\begin{aligned}\int xe^{(3x^2-1)}\,dx &= \frac{1}{6}\int e^{3x^2-1}\,6x\,dx\\ &= \frac{1}{6}\int e^u\,du = \frac{1}{6}e^u + C\\ &= \frac{1}{6}e^{3x^2-1}+C\end{aligned}$$

3.
$u = 4t^2-1$
$du = 8t\,dt$

$$\begin{aligned}\int \frac{t}{\sqrt[3]{4t^2-1}}\,dt &= \int t(4t^2-1)^{-1/3}\,dt\\ &= \frac{1}{8}\int (4t^2-1)^{-1/3}\,8t\,dt = \frac{1}{8}\int u^{-1/3}\,du\\ &= \frac{1}{8}\cdot\frac{3}{2}u^{2/3}+C = \frac{3}{16}(4t^2-1)^{2/3}+C\\ &= \frac{3}{16}\sqrt[3]{(4t^2-1)^2}+C\end{aligned}$$

4.
$u = 2x-1$; $x = \frac{u+1}{2}$
$du = 2dx$; $dx = \frac{du}{2}$

$$\begin{aligned}\int \frac{x}{2x-1}\,dx &= \int \frac{\frac{u+1}{2}}{u}\cdot\frac{du}{2}\\ &= \frac{1}{4}\int\frac{u+1}{u}\,du = \frac{1}{4}\int 1+\frac{1}{u}\,du\\ &= \frac{1}{4}(u+\ln|u|)+C = \frac{1}{4}(2x-1+\ln|2x-1|)+C\\ &= \frac{1}{4}(2x+\ln|2x-1|)+K\end{aligned}$$

5.
$u = x^2-1$
$du = 2x\,dx$

$$\begin{aligned}\int x(x+1)^3(x-1)^3\,dx &= \int x[(x+1)(x-1)]^3\,dx\\ &= \int x(x^2-1)^3\,dx = \frac{1}{2}\int (x^2-1)^3\,2x\,dx\\ &= \frac{1}{2}\int u^3\,du = \frac{1}{2}\cdot\frac{u^4}{4}+C\\ &= \frac{(x^2-1)^4}{8}+C\end{aligned}$$

6.

$$\begin{aligned}\int \frac{2t-1}{t}\,dt &= \int 2 - \frac{1}{t}\,dt \\ &= 2t - \ln|t| + C\end{aligned}$$

7.

$$\begin{aligned}\int \frac{(\ln x)^3}{3x}\,dx &= \frac{1}{3}\int (\ln x)^3 \cdot \frac{1}{x}\,dx \\ &= \frac{1}{3}\int u^3\,du = \frac{1}{3}\cdot\frac{u^4}{4} + C \\ &= \frac{(\ln x)^4}{12} + C\end{aligned}$$

$u = \ln x$

$du = \frac{1}{x}\,dx$