

SECTION 6.4 The Substitution Technique for Integration

IN-SECTION EXERCISES:

EXERCISE 1.

$$\begin{aligned}\frac{d}{dx} \left(\frac{(3-4x^2)^{101}}{101} \right) &= \frac{1}{101} \cdot 101(3-4x^2)^{101-1}(-8x) \\ &= (3-4x^2)^{100}(-8x)\end{aligned}$$

EXERCISE 2.

$$\begin{aligned}\int \overbrace{(3-4x^2)^{100}}^u \overbrace{(-8x) dx}^{du} &= \int u^{100} du && \text{(rewrite in terms of } u\text{)} \\ \mu = 3 - 4x^2 & && \\ du = -8x dx & && \\ &= \frac{u^{101}}{101} + C && \text{(integrate the 'new' problem)} \\ &= \frac{(3-4x^2)^{101}}{101} + C && \text{(transform back to } x\text{)}\end{aligned}$$

EXERCISE 3.

1.

$$\begin{aligned}\frac{d}{dx} \left(-\frac{1}{8} \cdot \frac{(3-4x^2)^{101}}{101} \right) &= -\frac{1}{8} \cdot \frac{1}{101} (101)(3-4x^2)^{100}(-8x) \\ &= (3-4x^2)^{100}x\end{aligned}$$

2. Linearity was used in going from

$$\int (3-4x^2)^{100} \left(\frac{-8}{-8} \right) x dx \quad \text{to} \quad \frac{1}{-8} \int \overbrace{(3-4x^2)^{100}}^u \overbrace{(-8x) dx}^{du} .$$

EXERCISE 4.

1. $\frac{du}{dx} = 3x^2$; it was noted that the variable part of this derivative, x^2 , also appeared as a factor in the integrand.

2.

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^3-1}} dx &= \frac{1}{3} \int \frac{3x^2}{\sqrt{x^3-1}} dx && \text{(multiply by 1 in form } \frac{3}{3}; \text{ linearity)} \\ &= \frac{1}{3} \int \frac{1}{\sqrt{u}} du && \text{(rewrite in terms of } u\text{)} \\ \mu = x^3 - 1 & && \\ du = 3x^2 dx & && \\ &= \frac{1}{3} \int u^{-1/2} du && \text{(rewrite with fractional exponents)} \\ &= \frac{1}{3} \cdot \frac{u^{1/2}}{1/2} + C && \text{(simple power rule)} \\ &= \frac{2}{3} \sqrt{x^3-1} + C && \text{(rewrite in terms of } x\text{)}\end{aligned}$$

3.

$$\begin{aligned}\frac{d}{dx} \left(\frac{2}{3} \sqrt{x^3 - 1} \right) &= \frac{d}{dx} \left(\frac{2}{3} (x^3 - 1)^{1/2} \right) \\ &= \frac{2}{3} \cdot \frac{1}{2} (x^3 - 1)^{-1/2} (3x^2) \\ &= \frac{x^2}{\sqrt{x^3 - 1}}\end{aligned}$$

EXERCISE 5.

1. The derivative of $y^2 + 2y + 1$ is $2y + 2 = 2(y + 1)$; it was noted that this derivative appeared (off only by a constant) as a factor in the integrand.

2.

$$\begin{aligned}\int \frac{x+1}{(x^2+2x+1)^3} dx &= \int \frac{(\frac{1}{2})(2)(x+1)}{(x^2+2x+1)^3} dx = \frac{1}{2} \int \frac{2x+2}{(x^2+2x+1)^3} dx \\ &= \frac{1}{2} \int \frac{1}{u^3} du = \frac{1}{2} \int u^{-3} du \\ &= \frac{1}{2} \cdot \frac{u^{-2}}{-2} + C = -\frac{1}{4u^2} + C \\ &= -\frac{1}{4(x^2+2x+1)^2} + C\end{aligned}$$

$u = x^2 + 2x + 1$
 $du = (2x + 2) dx$

3.

$$\begin{aligned}\frac{d}{dy} \left(-\frac{1}{4(y^2+2y+1)^2} \right) &= \frac{d}{dy} \left(-\frac{1}{4}(y^2+2y+1)^{-2} \right) \\ &= -\frac{1}{4} (-2)(y^2+2y+1)^{-3} (2y+2) \\ &= \frac{1}{2} (y^2+2y+1)^{-3} 2(y+1) \\ &= \frac{y+1}{(y^2+2y+1)^3}\end{aligned}$$

EXERCISE 6.

1. Define $g(x) = \ln|x|$, so that $g(f(x)) = \ln|f(x)|$.

Recall that $g'(x) = \frac{1}{x}$, for all $x \neq 0$. By the Chain Rule,

$$\frac{d}{dx} \ln|f(x)| = \frac{d}{dx} g(f(x)) = g'(f(x)) \cdot f'(x) = \frac{1}{f(x)} \cdot f'(x).$$

2.

$$f(0) = \frac{\ln|3 \cdot 0 + 5| + 3 - \ln 5}{3} = \frac{3}{3} = 1,$$

so the point $(0, 1)$ lies on the graph of f . Also,

$$f'(x) = \frac{d}{dx} \left(\frac{1}{3} (\ln|3x+5| + 3 - \ln 5) \right) = \frac{1}{3} \left(\frac{1}{3x+5} (3) + 0 \right) = \frac{1}{3x+5}.$$

END-OF-SECTION EXERCISES:

1.

$$\int (2x-1)^{17} dx = \int (2x-1)^{17} \frac{2}{2} dx = \frac{1}{2} \int (2x-1)^{17} (2 dx)$$

$$\begin{aligned} u &= 2x-1 \\ du &= 2 dx \end{aligned} \quad \begin{aligned} &= \frac{1}{2} \int u^{17} du = \frac{1}{2} \frac{u^{18}}{18} + C \\ &= \frac{1}{36} (2x-1)^{18} + C \end{aligned}$$

2.

$$\int 5t\sqrt{t^2+3} dt = 5 \int (t^2+3)^{1/2} \frac{2}{2} t dt = \frac{5}{2} \int (t^2+3)^{1/2} (2t dt)$$

$$\begin{aligned} u &= t^2+3 \\ du &= 2t dt \end{aligned} \quad \begin{aligned} &= \frac{5}{2} \int u^{1/2} du = \frac{5}{2} \frac{u^{3/2}}{3/2} + C \\ &= \frac{5}{3} (t^2+3)^{3/2} + C = \frac{5}{3} \sqrt{(t^2+3)^3} + C \end{aligned}$$

3.

$$\int \frac{3 \ln 4x}{x} dx = 3 \int \ln 4x \left(\frac{1}{x} dx\right) = 3 \int u du$$

$$\begin{aligned} u &= \ln 4x \\ du &= \frac{1}{4x} \cdot 4 dx \\ &= \frac{1}{x} dx \end{aligned} \quad \begin{aligned} &= 3 \frac{u^2}{2} + C = \frac{3}{2} (\ln 4x)^2 + C \end{aligned}$$

4.

$$\int (4e^{2t} + e^{1+t}) dt = 4 \int e^{2t} dt + \int e^{1+t} dt$$

$$\left\{ \int e^{kx} dx = \frac{1}{k} e^{kx} + C \right. \quad \begin{aligned} &= 4 \cdot \frac{1}{2} e^{2t} + \int e^u du \\ &= 2e^{2t} + e^{1+t} + C \end{aligned} \quad \begin{aligned} u &= 1+t \\ du &= dt \end{aligned}$$

5.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int \frac{e^{\sqrt{x}}}{\sqrt{x}} \frac{2}{2} dx = 2 \int e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} dx$$

$$\begin{aligned} u &= \sqrt{x} = x^{1/2} \\ du &= \frac{1}{2} x^{-1/2} dx \\ &= \frac{1}{2\sqrt{x}} dx \end{aligned} \quad \begin{aligned} &= 2 \int e^u du = 2e^u + C \\ &= 2e^{\sqrt{x}} + C \end{aligned}$$

Try re-doing this problem, taking $u = e^{\sqrt{x}}$.

6.

$$\int \frac{-1}{2u+5} du = - \int \frac{1}{2u+5} \cdot \frac{2}{2} du = -\frac{1}{2} \int \frac{1}{2u+5} (2 du)$$

$$\begin{aligned} w &= 2u+5 \\ dw &= 2 du \end{aligned} \quad \begin{aligned} &= -\frac{1}{2} \int \frac{1}{w} dw = -\frac{1}{2} \ln |w| + C \\ &= -\frac{1}{2} \ln |2u+5| + C \end{aligned}$$

7.

$$\int \frac{4t+2}{\sqrt{(t^2+t+1)^3}} dt = \int \frac{2(2t+1)}{(t^2+t+1)^{3/2}} dt = 2 \int (t^2+t+1)^{-3/2} (2t+1) dt$$

$$= 2 \int u^{-3/2} du = 2 \cdot \frac{u^{-1/2}}{-1/2} + C$$

$$= -4(t^2+t+1)^{-1/2} + C = \frac{-4}{\sqrt{t^2+t+1}} + C$$

$u = t^2 + t + 1$
 $du = (2t + 1) dt$

8.

$$\int (e^x + 1)^5 \cdot 3e^x dx = 3 \int (e^x + 1)^5 e^x dx = 3 \int u^5 du$$

$$= 3 \frac{u^6}{6} + C = \frac{1}{2} (e^x + 1)^6 + C$$

$u = e^x + 1$
 $du = e^x dx$

9. First, find *all* functions f with the specified derivative:

$$f(x) = \int e^x (e^x + 1)^3 dx = \int u^3 du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{(e^x + 1)^4}{4} + C$$

$u = e^x + 1$
 $du = e^x dx$

Then, for the graph of f to pass through the point $(0, 4)$, it must be that $f(0) = 4$:

$$f(0) = 4 \iff \frac{(e^0 + 1)^4}{4} + C = 4$$

$$\iff 4 + C = 4 \iff C = 0$$

Take $f(x) = \frac{(e^x + 1)^4}{4}$.10. Since $d'(t) = v(t)$, the distance function is an antiderivative of the velocity function. First, find *all* antiderivatives:

$$d(t) = \int (t-2)^3 dt = \int u^3 du = \frac{u^4}{4} + C = \frac{(t-2)^4}{4} + C$$

$$u = t - 2$$

$$du = dt$$

Then, knowing that $d(1) = \frac{1}{2}$ yields

$$\frac{(1-2)^4}{4} + C = \frac{1}{2} \iff C = \frac{1}{4}$$

Take $d(t) = \frac{(t-2)^4}{4} + \frac{1}{4}$.

11. a) The student pulled a *variable* out of the integral in going from

$$\int \frac{2x}{2x} (x^2 + 1)^5 dx \quad \text{TO} \quad \frac{1}{2x} \int (x^2 + 1)^5 (2x dx) ,$$

which is NOT allowed.

- b) The student's 'solution' is NOT correct:

$$\begin{aligned} \frac{d}{dx} \left(\frac{(x^2 + 1)^6}{12x} \right) &= \frac{(12x)6(x^2 + 1)^5(2x) - (x^2 + 1)^6(12)}{(12x)^2} \\ &= \frac{12(x^2 + 1)^5[12x^2 - (x^2 + 1)]}{144x^2} \\ &= \frac{(x^2 + 1)^5(11x^2 - 1)}{12x^2} , \end{aligned}$$

which is NOT equal to $(x^2 + 1)^5$. (For example, substitute $x = 1$ into both formulae, to see that they are NOT the same.)