SECTION 6.3 Analyzing a Falling Object (Optional)

IN-SECTION EXERCISES:

EXERCISE 1.

1. The particle starts at position 0 at $t = 0$, and moves to the right at a uniform speed of 3 feet per second.
2. The particle starts at position 0 at $t = 0$, and moves to the left at a uniform speed of 3 feet per second.
3. The particle starts at position 0 at $t = 0$, and moves to the left, continually picking up speed as it travels.
4. The particle starts at position 2 at $t = 0$. It travels to the left at a constant speed of 2 units per second for the first second; so at $t = 1$ it is at position 0. Then, it turns around, and travels to the right at a constant speed of 2 units per second.

EXERCISE 2.

In all cases, the results agree with the description of motion given in Exercise 1.

1. For $d(t) = 3t$, $v(t) = d'(t) = 3$.
2. For $d(t) = -3t$, $v(t) = d'(t) = -3$.
3. For $d(t) = -t^2$, $v(t) = d'(t) = -2t$.
4. For $d(t) = 2|t-1|$, first give a piecewise description of d:

   $$d(t) = \begin{cases} 2(t-1) & \text{for } t \geq 1 \\ 2(1-t) & \text{for } t < 1 \end{cases}$$

   Thus:

   $$v(t) = d'(t) = \begin{cases} 2 & \text{for } t > 1 \\ -2 & \text{for } t < 1 \end{cases}$$

   The function d is not differentiable at $t = 1$.

EXERCISE 3.

$d(t) = t^3 - 2t^2 + 3$; $v(t) = d'(t) = 3t^2 - 4t$

Position at:

$t = 1$: $d(1) = 1^3 - 2(1^2) + 3 = 2$ meters
$t = -1$: $d(-1) = (-1)^3 - 2(-1)^2 + 3 = 0$ meters
$t = 0$: $d(0) = 0^3 - 2(0^2) + 3 = 3$ meters
$t = T$: $d(T) = T^3 - 2T^2 + 3$ meters

Velocity at:

$t = 1$: $v(1) = 3(1^2) - 4(1) = -1$ meters/second
$t = -1$: $v(-1) = 3(-1)^2 - 4(-1) = 7$ meters/second
$t = 0$: $v(0) = 3(0^2) - 4(0) = 0$ meters/second
$t = T$: $v(T) = 3T^2 - 4T$ meters/second

Speed at:

$t = 1$: $|v(1)| = |-1| = 1$ meters/second
$t = -1$: $|v(-1)| = 7$ meters/second
$t = 0$: $|v(0)| = 0$ meters/second
$t = T$: $|v(T)| = |3T^2 - 4T|$ meters/second

EXERCISE 4.

1. For $d(t) = 3t$, $v(t) = d'(t) = 3$, and $a(t) = v'(t) = 0$. The velocity does not change as the particle moves.

   For $d(t) = -3t$, $v(t) = d'(t) = -3$, and $a(t) = v'(t) = 0$. The velocity does not change as the particle moves.

   For $d(t) = -t^2$, $v(t) = d'(t) = -2t$, and $a(t) = v'(t) = -2$.

   For $d(t) = 2|t-1|$, it was found that

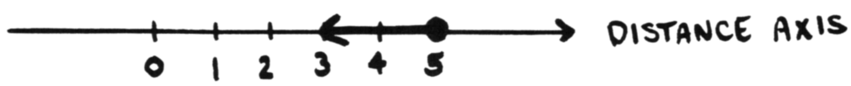
$$v(t) = d'(t) = \begin{cases} 2 & \text{for } t > 1 \\ -2 & \text{for } t < 1 \ . \end{cases}$$

   Thus, $a(t) = v'(t) = 0$ for $t \neq 1$.

2. $v(t) = d'(t) = 6t^2 + 2t - 3$; $a(t) = v'(t) = d''(t) = 12t + 2$.

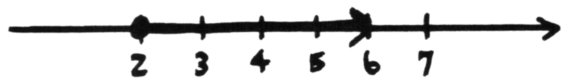
EXERCISE 5.

1.



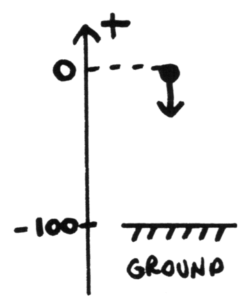
2.

3. $v(t) = d'(t) = 2t$; $v(2) = 2(2) = 4$ ft/sec



EXERCISE 6.

Since 'up' is the positive direction, and the force acting on the object acts DOWN, it enters the equation with a minus sign:



$$\begin{aligned} -mg = m \cdot a(t) \quad &\implies \quad v'(t) = -g \implies \quad v(t) = -gt + v_0 \\ &\implies \quad v(t) = -gt \implies \quad d'(t) = -gt \\ &\implies \quad d(t) = -\frac{gt^2}{2} + d_0 \implies \quad d(t) = -\frac{gt^2}{2} \end{aligned}$$

The object hits the ground at time t for which $d(t) = -100$; that is,

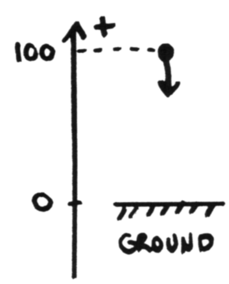
$$-\frac{gt^2}{2} = -100 \ .$$

This happens when $t = \sqrt{\frac{200}{g}} \approx 2.5$ seconds.

The velocity function is $v(t) = -gt$, so $v(2.5) \approx -80$ ft/sec. The minus sign indicates that the velocity points in the negative direction, which, for the choices made, is down.

EXERCISE 7.

Again, the force enters the equation with a minus sign attached. Note that $d(0) = 100$.



$$\begin{aligned} -mg = m \cdot a(t) \quad &\implies \quad v'(t) = -g \implies \quad v(t) = -gt + v_0 \\ &\implies \quad v(t) = -gt \implies \quad d'(t) = -gt \\ &\implies \quad d(t) = -\frac{gt^2}{2} + d_0 \implies \quad d(t) = -\frac{gt^2}{2} + 100 \end{aligned}$$

The object hits the ground at time t for which $d(t) = 0$; that is,

$$-\frac{gt^2}{2} + 100 = 0 \ .$$

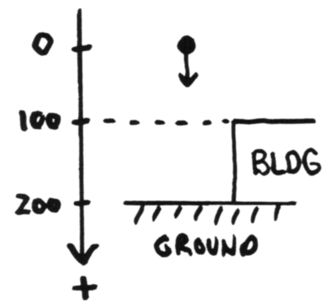
This happens when $t = \sqrt{\frac{200}{g}} \approx 2.5$ seconds.

The velocity function is $v(t) = -gt$, so $v(2.5) \approx -80$ ft/sec.

EXERCISE 8.

1. Choose 'down' as the positive direction; let '0' coincide with the initial position of the falling object. Then:

$$\begin{aligned} mg = ma(t) &\implies a(t) = g \implies v'(t) = g \\ &\implies v(t) = gt + v_0 \implies v(t) = gt \\ &\implies d'(t) = gt \implies d(t) = g\frac{t^2}{2} + d_0 \\ &\implies d(t) = g\frac{t^2}{2} \end{aligned}$$



2. The object hits the ground when $d(t) = 200$:

$$g\frac{t^2}{2} = 200 \quad \iff \quad t^2 = \frac{400}{g} \quad \iff \quad t = \pm\sqrt{\frac{400}{g}}$$

Choosing the nonnegative answer and approximating, $t \approx \sqrt{\frac{400}{32}} = 3.54$ seconds.

3. $d(1) = g\frac{1^2}{2} = g/2$. After 1 second, the object is approximately $200 - \frac{1}{2} \cdot 32 = 184$ feet above the ground.
   $d(2) = g\frac{2^2}{2} = 2g$. After 2 seconds, the object is approximately $200 - 2(32) = 136$ feet above the ground.
4. The object reaches the top of the building when $d(t) = 100$:

$$g\frac{t^2}{2} = 100 \quad \iff \quad t^2 = \frac{200}{g} \quad \iff \quad t = \pm\sqrt{\frac{200}{g}}$$

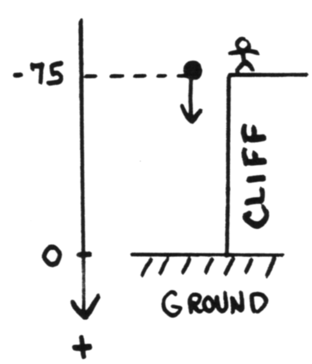
Approximating, $t \approx 2.5$ seconds.

4. $v(1) = g \cdot 1 \approx 32$ ft/second.
   $v(2) = g \cdot 2 \approx 64$ ft/second.
   $v(3.54) = g \cdot 3.54 \approx 113.28$ ft/second.
5. The equation of motion is only valid until the object hits the ground. So, it is only valid for approximately 3.54 seconds.

END-OF-SECTION EXERCISES:

1. 'Down' is chosen as the positive direction; '0' coincides with the ground.

$$\begin{aligned} mg = ma(t) &\implies a(t) = g \implies v'(t) = g \\ &\implies v(t) = gt + v_0 \implies v(t) = gt - 20 \\ &\implies d'(t) = gt - 20 \implies d(t) = g\frac{t^2}{2} - 20t + d_0 \\ &\implies d(t) = g\frac{t^2}{2} - 20t - 75 \end{aligned}$$



2. $v(t) = gt - 20$

3. $v(t) = 0 \iff gt - 20 = 0 \iff t = \frac{20}{g}$.
   It will go up for approximately 0.63 seconds, before it starts to come down.
4. $d(.63) = g\frac{(.63)^2}{2} - 20(.63) - 75 \approx -81.25$ feet. Thus, the object reaches a maximum height of approximately 81.25 feet.
5. Set $d(t)$ equal to -75 and solve for t:

$$\begin{aligned} g\frac{t^2}{2} - 20t - 75 = -75 \quad &\iff \quad g\frac{t^2}{2} - 20t = 0 \\ &\iff \quad t(\frac{gt}{2} - 20) = 0 \\ &\iff \quad t = 0 \ \text{ or } \ t = \frac{40}{g} \end{aligned}$$

   The object passes the person who threw it in approximately 1.25 seconds. (It takes just as long to go UP as it does to come back DOWN to the person!)
6. Set $d(t) = 0$ and solve for t:

$$g\frac{t^2}{2} - 20t - 75 = 0 \quad \iff \quad t = \frac{20 \pm \sqrt{400 - 4(\frac{g}{2})(-75)}}{g}$$

   Choosing the positive answer, it takes approximately 2.88 seconds for the object to hit the ground.