SECTION 5.6 Asymptotes—Checking Behavior at Infinity

IN-SECTION EXERCISES:

EXERCISE 1.

The limit statement

$$\lim_{x\to c^-} f(x) = \infty$$

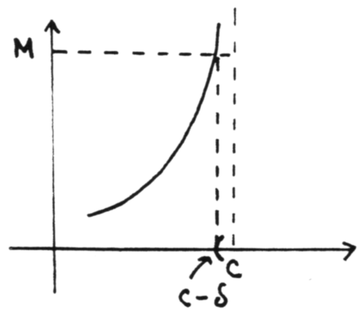
means that $f(x)$ can be made as large and positive as desired, by requiring that x be sufficiently close to c (and less than c).

Precisely,

$$\lim_{x\to c^-} f(x) = \infty \quad \iff \quad \begin{array}{l} \forall\ M > 0,\ \exists\ \delta > 0 \text{ such that if} \\ x \in (c-\delta, c), \text{ then } f(x) > M\ . \end{array}$$

The sentence $\lim_{x\to c^-} f(x) = \infty$ can also be written:

$$\text{As } \ x \to c^-, \ \ f(x) \to \infty.$$



EXERCISE 2.

The limit statement

$$\lim_{x\to c^+} f(x) = -\infty$$

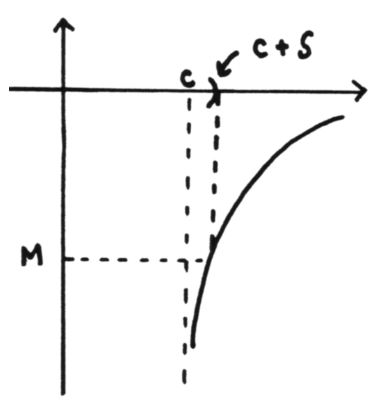
means that $f(x)$ can be made as large and negative as desired, by requiring that x be sufficiently close to c (and greater than c).

Precisely,

$$\lim_{x\to c^+} f(x) = -\infty \quad \iff \quad \begin{array}{l} \forall\ M < 0,\ \exists\ \delta > 0 \text{ such that if} \\ x \in (c, c+\delta), \text{ then } f(x) < M\ . \end{array}$$

The sentence $\lim_{x\to c^+} f(x) = -\infty$ can also be written:

$$\text{As } \ x \to c^+, \ \ f(x) \to -\infty.$$



EXERCISE 3.
PRECISE SOLUTION:

Multiply by 1 in an appropriate form:

$$f(x) = \frac{5x^3}{2x(x^2-1)} = \frac{5x^3}{2x^3-2x} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \frac{5}{2-\frac{2}{x^2}} \ .$$

Then,

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{5}{2-\frac{2}{x^2}} = \frac{\lim_{x\to\pm\infty} 5}{\lim_{x\to\pm\infty}(2-\frac{2}{x^2})} = \frac{5}{2} \ .$$

Thus, $y = \frac{5}{2}$ is a horizontal asymptote for f.

ABBREVIATED SOLUTION:
For large values of x,

$$f(x) \approx \frac{5x^3}{2x^3} = \frac{5}{2} \ .$$

Thus, $y = \frac{5}{2}$ is a horizontal asymptote for f.

EXERCISE 4.
The limit statement

$$\lim_{x\to-\infty} f(x) = L$$

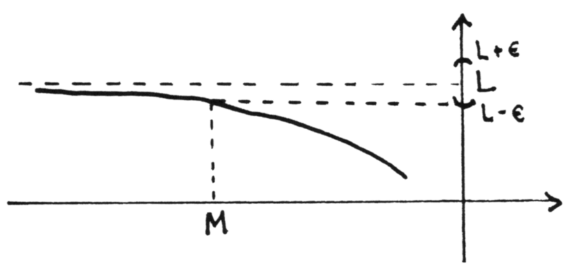
means that the numbers $f(x)$ can be made as close to L as desired, by requiring that x be sufficiently large and negative.

Precisely,

$$\lim_{x\to-\infty} f(x) = L \quad \iff \quad \begin{array}{l} \forall\ \epsilon > 0,\ \exists\ M < 0 \text{ such that if } x < M, \\ \text{then } |f(x) - L| < \epsilon \ . \end{array}$$

The sentence $\lim_{x\to-\infty} f(x) = L$ can also be written:

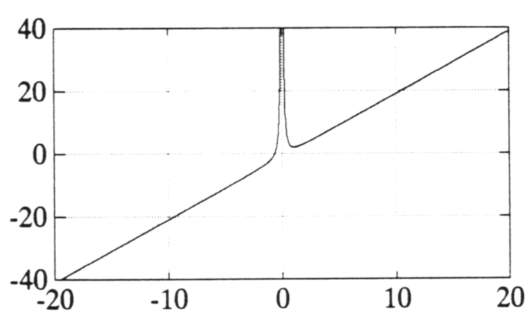
$$\text{As } \ x \to -\infty, \ \ f(x) \to L.$$



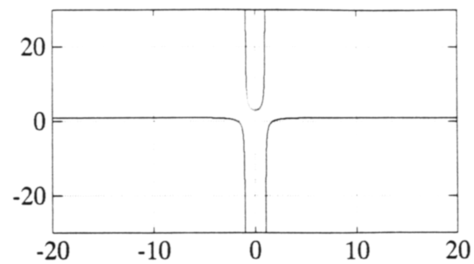
EXERCISE 5.

1. vertical asymptote: $x = 0$

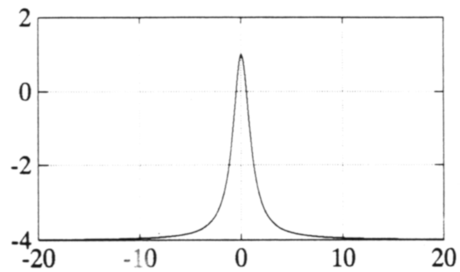
   oblique asymptote: $y = 2x - 1$



2. vertical asymptotes: $x = 1$ and $x = -1$
   horizontal asymptote: $y = 1$



3. horizontal asymptote: $y = -4$



EXERCISE 6.
Note first that $\mathcal{D}(f) = \mathbb{R} - \{1, -1\}$. However, (in the same breath), you must notice that the numerator is also zero when $x = 1$ and when $x = -1$. Indeed,

$$x^3 + 2x^2 - x - 2 = (x-1)(x+1)(x+2) \ .$$

Thus, for $x \neq 1$ and $x \neq -1$,

$$f(x) = \frac{(x-1)(x+1)(x+2)}{(x-1)(x+1)} = x + 2 \ .$$

The graph of f is the same as the line $y = x + 2$, punctured at $x = 1$ and $x = -1$.