SECTION 5.4 Graphing Functions—Some Basic Techniques

IN-SECTION EXERCISES:

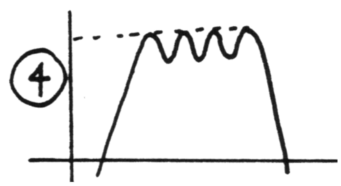
EXERCISE 1.
FIRST GRAPH:
global minimum value: -3
global maximum value: none
global minimum point: $(2, -3)$

SECOND GRAPH:
global minimum value: -1
global maximum value: 4
global minimum points: $\{(x, -1) \mid x \in [1, 3]\}$
global maximum point: $(-1, 4)$

EXERCISE 2.

1. FALSE. Not every local maximum is a global maximum. See the sketch below.

2. TRUE. Every global maximum point is a local maximum point.
3. TRUE.
4. FALSE. There may be many places where the greatest function value is attained. See the sketch below.



EXERCISE 3.
The first graph cannot be correct, since P must have a horizontal tangent line at $(1, 0)$.

The second graph cannot be correct, since P must have a horizontal tangent line at $(1, 0)$, and not further to the right.

EXERCISE 4.

1. $\mathcal{D}(P) = \mathbb{R}$. Plot a few points:

   $P(0) = 1,\ \ P(1) = -4,\ \ P(-1) = -12$

   $P'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x+2)(x-1)$

   $P'(x) = 0 \iff x = 0$ or $x = -2$ or $x = 1$

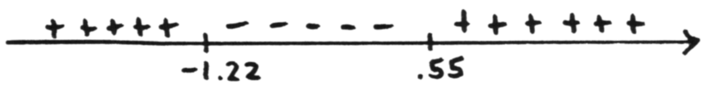
   $P''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2)$

   $P''(x) = 0 \iff x = \frac{-2\pm\sqrt{4-4(3)(-2)}}{2\cdot 3}$

   $\frac{-2+\sqrt{4-4(3)(-2)}}{2\cdot 3} \approx 0.55$

   $\frac{-2-\sqrt{4-4(3)(-2)}}{2\cdot 3} \approx -1.22$

   Sign of $P''(x)$:

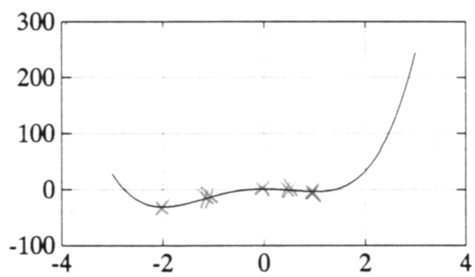


   Thus, from left to right, P is concave up, down, up.

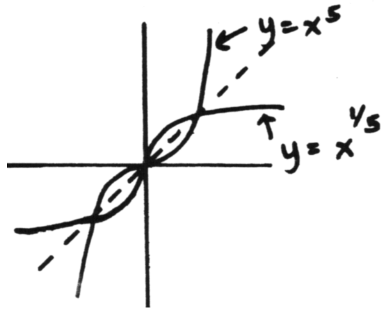
   For large x, $P(x) \approx 3x^4$; so as $x \to \pm\infty$, $P(x) \to \infty$.

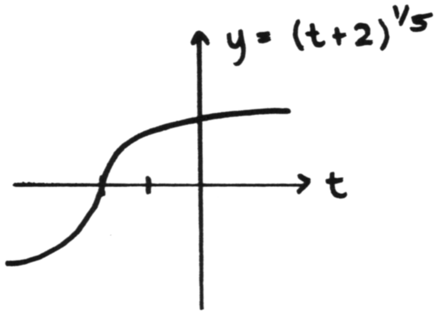
   Details: The two x-axis intercepts can be 'zeroed in on', using the Intermediate Value Theorem, if desired.

   A MATLAB graph is given.



2. Here's one approach, that doesn't use calculus. The graph of f is the same as the graph of $y = x^{1/5}$, except shifted 2 units to the left. And, $y = x^{1/5}$ is the inverse of $y = x^5$, so its graph is a reflection about the line $y = x$. Just 'build the graph up'!





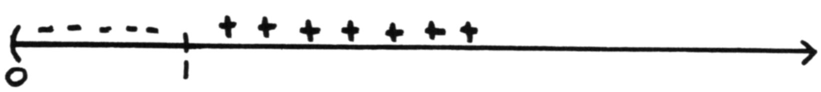
3. $\mathcal{D}(f) = \mathbb{R} - \{1, -1\}$.

g is an even function, so the graph is symmetric about the x-axis; and for $x > 0$, $g(x) = \frac{x}{x^2-1}$.
Plot a few points: $g(0) = 0$, $g(2) = \frac{2}{3}$, $g(\frac{1}{2}) = -\frac{2}{3}$.
As x approaches 1 from the right, $g(x)$ approaches $-\infty$;
as x approaches 1 from the left, $g(x)$ approaches ∞.
(More on limits involving infinity in section 5.6.)
$g'(x) = \frac{(x^2-1)(1)-x(2x)}{(x^2-1)^2} = -\frac{x^2+1}{(x^2-1)^2}$.
Remember that this formula is only valid for $x \geq 0$. Note that $g'(x)$ is never equal to zero. Also, as x approaches 0 from the right, $g'(x)$ approaches -1. By symmetry, as x approaches 0 from the left, $g'(x)$ approaches 1. There is a 'kink' at 0. (This 'comes from' the absolute value curve.)

Computing the second derivative,

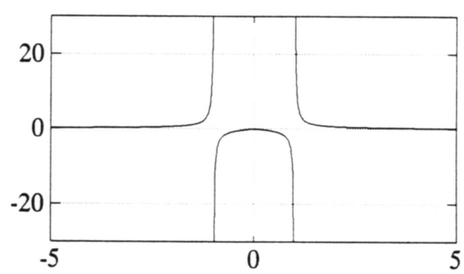
$$\begin{aligned} g''(x) &= \frac{(x^2-1)^2(-2x) + (x^2+1)2(x^2-1)(2x)}{(x^2-1)^4} \\ &= \frac{2x(x^2-1)[-(x^2-1) + 2(x^2+1)]}{(x^2-1)^4} \\ &= \frac{2x[x^2+3]}{(x^2-1)^3} \ ; \end{aligned}$$

again, this formula is only valid for $x \geq 0$. Note that $g''(x) = 0$ only when $x = 0$, and g'' has a discontinuity at $x = 1$. Computing the sign of $g''(x)$ for $x > 0$:

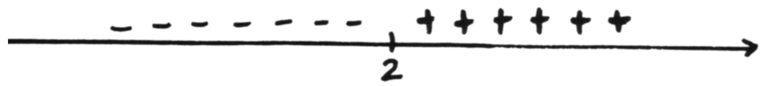


Thus, g is concave down on $(0, 1)$ and concave up on $(1, \infty)$.

Details: As $x \to \infty$, $g(x) \to 0$.
A MATLAB graph is given:



4. $\mathcal{D}(f) = \mathbb{R}$
$f(0) = 0;\ \ f(1) = \frac{1}{e};\ \ f(-1) = -e$
$f'(x) = x(-e^{-x}) + (1)e^{-x} = e^{-x}(1-x)$
$f'(x) = 0 \iff x = 1$; the point $(1, \frac{1}{e})$ is a critical point.
$f''(x) = e^{-x}(-1) + (-e^{-x})(1-x) = -e^{-x}[1+1-x] = -e^{-x}(2-x)$
$f''(x) = 0 \iff x = 2$; the point $(2, \frac{2}{e^2})$ is a candidate for an inflection point.
Sign of $f''(x)$:



Thus, f is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$.

Details: Note that for $x > 0$, $f(x) > 0$. Also, as $x \to \infty$, $f(x) \to 0$ (since e^x gets bigger much faster than x).

$(1, \frac{1}{e})$ is a local and global maximum point.
There are no local or global minima.
$(2, \frac{2}{e^2})$ is an inflection point.
$(0, 0)$ is the only x-axis intercept, and the only y-axis intercept.
The graph is concave up on $(2, \infty)$ and concave down on $(-\infty, 2)$.
The graph increases on $(-\infty, 1)$ and decreases on $(1, \infty)$.

Here's a MATLAB graph: