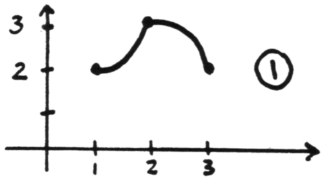
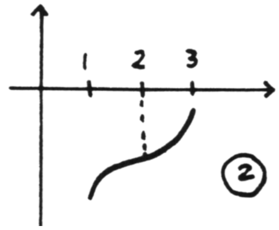
SECTION 5.3 The Second Derivative—Inflection Points

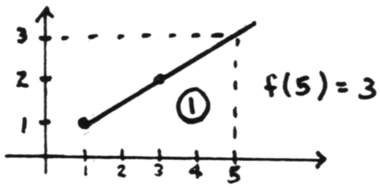
IN-SECTION EXERCISES:

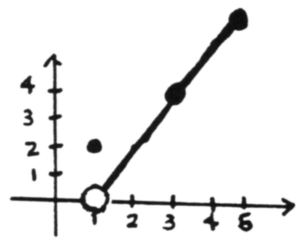
EXERCISE 1.





EXERCISE 2.





EXERCISE 3.

1. If $x = 2$, then $x^2 = 4$; true.

   Converse: If $x^2 = 4$, then $x = 2$; false. (Take $x = -2$. Then the hypothesis '$(-2)^2 = 4$' is true, but the conclusion '$-2 = 2$' is false.)
2. $1 = 2 \implies 1 + 1 = 2$; (vacuously) true.

   Converse: $1 + 1 = 2 \implies 1 = 2$; false.
3. If $1 = 2$, then $2 = 3$; (vacuously) true.

   Converse: If $2 = 3$, then $1 = 2$; (vacuously) true.
4. $A \Rightarrow B$ has converse $B \Rightarrow A$ which has converse $A \Rightarrow B$. Thus, the converse of the converse is the original implication.

   original implication: $A \Rightarrow B$;
   converse: $B \Rightarrow A$;
   contrapositive of the converse: not $A \Rightarrow$ not B.

   original implication: $A \Rightarrow B$;
   contrapositive: not $B \Rightarrow$ not A;
   converse of the contrapositive: not $A \Rightarrow$ not B.

EXERCISE 4.

1. $f'(x) = 4x$; $f''(x) = 4$. When x changes by an amount Δx, the slopes of the tangent lines should change by four times this amount.
2. At $x + \Delta x$, the slope of the tangent line is $f'(x + \Delta x) = 4(x + \Delta x) = 4x + 4\Delta x$. At x, the slope of the tangent line is $f'(x)$.
3. $$\begin{aligned} \Delta f' &= f'(x + \Delta x) - f'(x) \\ &= (4x + 4\Delta x) - 4x \\ &= 4\Delta x \ . \end{aligned}$$

   Thus, the slopes of the tangent lines have indeed changed by four times the amount that x has changed.

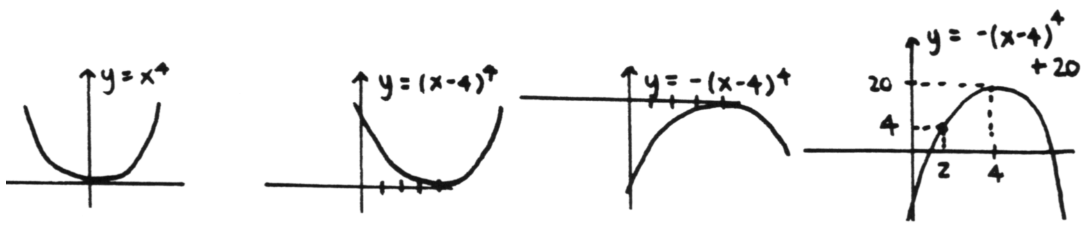
EXERCISE 5.

1. $f'(x) = 3x^2$; $f''(x) = 6x$. At the point $(2, 8)$, the slopes of the tangent lines are changing $f''(2) = 6 \cdot 2 = 12$ times as fast as x changes.
2. In moving from $x = 2$ to $x = 2.1$, the change in x is 0.1; thus, the expected change in the slopes is $12 \cdot 0.1 = 1.2$.

3. $f'(2) = 3(2)^2 = 12$; $f'(2.1) = 3(2.1)^2 = 13.23$. Thus, $\Delta f' = 13.23 - 12 = 1.23$.
4. The estimate was a bit low. This is because, as soon as we move away from the point $(2, 8)$, the rate of change of the slopes is actually *greater than* 12.

EXERCISE 6.

1. Start with $y = x^4$; shift 4 to the right; make all the y-values negative; shift up 20.



2. $f(2) = -(2-4)^4 + 20 = 4$
3. $f'(x) = -4(x-4)^3$; $f''(x) = -12(x-4)^2$. Thus, $f''(2) = -12(2-4)^2 = -48$. When x changes by a small amount, we expect the slopes of the tangent lines to change by -48 times this amount.
4. When x changes by 0.1, the slopes should change by approximately $-48 \cdot (0.1) = -4.8$.
5. $f'(2) = -4(2-4)^3 = 32$; $f'(2.1) = -4(2.1-4)^3 = 27.436$. Note that

$$f'(2.1) - f'(2) = 27.436 - 32 = -4.564 \ .$$

Compare this with the 'expected' change of -4.8.

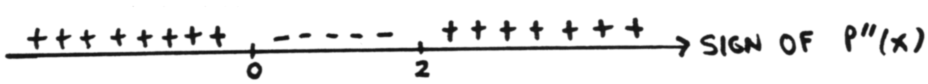
EXERCISE 7.

1. The domain of P is $\mathbb{R}$.

$$\begin{gathered} P'(x) = 4x^3 - 12x^2 - 7 \\ P''(x) = 12x^2 - 24x = 12x(x-2) \ ; \\ P''(x) = 0 \iff x = 0 \text{ or } x = 2 \ . \end{gathered}$$

When $x = 0$, $P(0) = 1$, so $(0, 1)$ is a candidate for an inflection point.
When $x = 2$, $P(2) = -29$, so $(2, -29)$ is a candidate for an inflection point.

Determine the sign of P'' everywhere:
Test Points: $P''(-1) = (-)(-) > 0$, $P''(1) = (+)(-) < 0$, $P''(3) = (+)(+) > 0$.



Thus, both candidates are indeed inflection points, since the concavity of the function *changes* as we pass through each point.

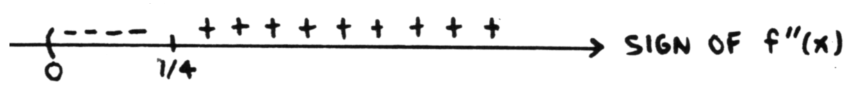
2. The domain of f is $(0,\infty)$.

$$
\begin{aligned}
f'(x) &= \frac{1}{2}x^{-1/2} + 2x \\
f''(x) &= -\frac{1}{4}x^{-3/2} + 2 \\
&= -\frac{1}{4\sqrt{x^3}} + 2\cdot\frac{4\sqrt{x^3}}{4\sqrt{x^3}} \\
&= \frac{-1+8\sqrt{x^3}}{4\sqrt{x^3}}
\end{aligned}
$$

Note that

$$f''(x)=0 \iff -1+8\sqrt{x^3}=0 \iff \sqrt{x^3}=\frac{1}{8} \iff x^3=\frac{1}{64} \iff x=\frac{1}{4}\ .$$

The point $(\frac14, f(\frac14)) = (\frac14, \frac{9}{16})$ is the only candidate for an inflection point.
Determine the sign of f'' everywhere:
Test Points: $f''(\frac18) < 0$, $f''(1) > 0$.



Thus, $(\frac14, \frac{9}{16})$ is indeed an inflection point.

EXERCISE 8.
Some approximation is necessary. It is assumed that the patterns displayed at the graph boundaries continue.

1. The function is positive on the interval $(-1.75,\infty)$.
   The function is negative on $(-\infty,-1.75)$.
2. The function increases on $(-\infty,1)\cup(2.7,\infty)$.
   The function decreases on $(1,2.7)$.
3. It is somewhat difficult to tell where the concavity changes, without further information. Thus, the following answers are certainly approximate:

   The function is concave up on $(-1,0)\cup(2,\infty)$.
   The function is concave down on $(-\infty,-1)\cup(0,2)$.

EXERCISE 9.

1. **Proof**. Suppose that $f'(c)=0$ and $f''(c)<0$. Assume, for simplicity, that f is defined on both sides of c.

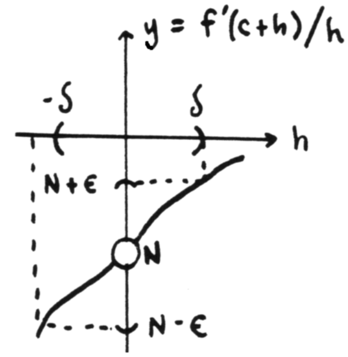
   Recall that $f''=(f')'$. Thus, $f''(c)<0$ means that the limit

$$\lim_{h\to 0}\frac{f'(c+h)-f'(c)}{h} = \lim_{h\to 0}\frac{f'(c+h)-0}{h}$$

   exists, and is negative. Call the value of this limit N (for 'negative'). Thus, it is possible to get the values $\frac{f'(c+h)}{h}$ as close to N as desired, merely by requiring that h be sufficiently close to 0. Remember that when h is close to 0, $c+h$ is close to c. In particular, when $h<0$, $c+h$ is to the left of c; and when $h>0$, $c+h$ is to the right of c.

Refer to the sketch. Choose ϵ so that every number in the interval $I := (N-\epsilon, N+\epsilon)$ is negative. Then, find δ so that whenever h is within δ of 0, the numbers $\frac{f'(c+h)}{h}$ end up in I.
If $h < 0$, and within δ of 0, then multiplying both sides of the inequality

$$\frac{f'(c+h)}{h} < 0$$



by the negative number h yields

$$f'(c+h) > 0 \ ,$$

so the function is increasing to the left of the point $(c, f(c))$.
Similarly, if $h > 0$ and within δ of 0, then we get

$$f'(c+h) < 0 \ ,$$

so the function is decreasing to the right of the point $(c, f(c))$.
By the First Derivative Test, the point $(c, f(c))$ is a local maximum. ∎

2.

$$P'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2+x-2) = 12x(x+2)(x-1)$$
$$P''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2)$$

Now,

$$P'(x) = 0 \iff x = 0 \text{ or } x = -2 \text{ or } x = 1 \ ,$$

so there is a horizontal tangent line at each of these points.
$P''(0) = 12(0+0-2) < 0$, so P is concave down at $x = 0$. Thus, $(0, P(0))$ is a local maximum.
$P''(-2) = 12(3(-2)^2 + 2(-2) - 2) = 12(12-4-2) > 0$, so P is concave up at $x = -2$. Thus, $(-2, P(-2))$ is a local minimum.
$P''(1) = 12(3+2-2) > 0$, so P is also concave up at $x = 1$. Thus, $(1, P(1))$ is a local minimum.

END-OF-SECTION EXERCISES:

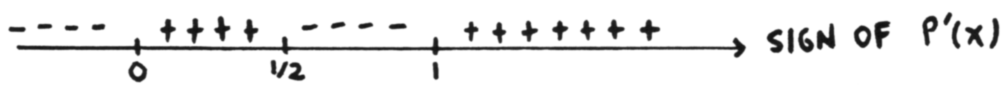
1.

$$P'(x) = 4x^3 - 6x^2 + 2x = 2x(2x^2 - 3x + 1) = 2x(2x-1)(x-1)$$
$$P''(x) = 12x^2 - 12x + 2 = 2(6x^2 - 6x + 1)$$

The critical points are summarized in the table below.

C	P(C)	WHY?	LOCAL EXT?
0	10	P'(C)=0	MIN
1/2	161/16	P'(C)=0	MAX
1	10	P'(C)=0	MIN

First Derivative Test: The sign of $P'(x)$ is given below:



Since P decreases to the left of $x = 0$ and increases to the right, there is a local minimum at $x = 0$.
Since P increases to the left of $x = \frac{1}{2}$ and decreases to the right, there is a local maximum at $x = \frac{1}{2}$.
Since P decreases to the left of $x = 1$ and increases to the right, there is a local minimum at $x = 1$.

Second Derivative Test: $P''(0) > 0$, so there is a local minimum at $x = 0$.
$P''(\frac{1}{2}) = 2(6(\frac{1}{2})^2 - 6(\frac{1}{2}) + 1) < 0$, so there is a local maximum at $x = \frac{1}{2}$.
$P''(1) > 0$, so there is a local minimum at $x = 1$.

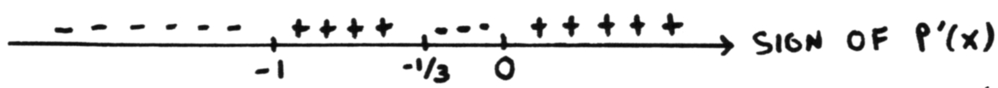
2.

$$
\begin{aligned}
P'(x) &= 36x^3 + 48x^2 + 12x = 12x(3x^2 + 4x + 1) = 12x(3x + 1)(x + 1) \\
P''(x) &= 108x^2 + 96x + 12
\end{aligned}
$$

The critical points are summarized in the table below.

C	P(C)	WHY?	LOCAL EXT
0	1	$P'(c) = 0$	
$-1/3$	$32/27$	$P'(c) = 0$	
-1	0	$P'(c) = 0$	

First Derivative Test: The sign of $P'(x)$ is given below:



Since P decreases to the left of $x = -1$ and increases to the right, there is a local minimum at $x = -1$.
Since P increases to the left of $x = -\frac{1}{3}$ and decreases to the right, there is a local maximum at $x = -\frac{1}{3}$.
Since P decreases to the left of $x = 0$ and increases to the right, there is a local minimum at $x = 0$.

Second Derivative Test: $P''(-1) > 0$, so there is a local minimum at $x = -1$.
$P''(-\frac{1}{3}) < 0$, so there is a local maximum at $x = -\frac{1}{3}$.
$P''(0) > 0$, so there is a local minimum at $x = 0$.

3. $f(x)$ is positive on $(-\infty, -2.5) \cup (-2, \infty)$.
   $f(x)$ is negative on $(-2.5, -2)$.
4. f increases on $(-4, -3) \cup (0, \infty)$.
   f decreases on $(-3, -2) \cup (-2, 0)$.
5. f is concave up on $(-2, 2)$.
   f is concave down on $(-3, -2) \cup (2, \infty)$.
6. $\mathcal{D}(f) = \mathbb{R} - \{-2\} = (-\infty, -2) \cup (-2, \infty)$.
7. $\mathcal{D}(f') = \mathbb{R} - \{-4, -3, -2\}$.
8. $\{x \mid f'(x) = 0\} = (-\infty, -4) \cup \{0\}$.
9. $\{x \mid f(x) > 10\} = (-2, -1.5)$.
10. $\{x \mid f'(x) > 0\} = (-4, -3) \cup (0, \infty)$.
11. $\{x \mid f''(x) < 0\} = (-3, -2) \cup (2, \infty)$.
12. $\lim_{x\to 0} f(x) = 2$.
13. $\lim_{t\to -2} f(t)$ does not exist.
14. $\lim_{y\to -4} f(y) = 4$
15. The critical points are: $\{(x, 4) \mid x \in (-\infty, -4)\}$, since $f'(x) = 0$ for all these points;
    $(0, 2)$, since $f'(0) = 0$;
    $(-4, 4)$ and $(-3, 8)$, since the derivative does not exist at these points.

16. $f(0) = 2$
    $f'(0) = 0$
    $f(1000) \approx 10$
    $f'(1000) \approx 0$
17. $\{x \in \mathcal{D}(f) \mid f \text{ is not differentiable at } x\} = \{-4, -3\}$.
18. The only inflection point is $(2, 6)$.
19. $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = f'(0) = 0$.
20. $\lim_{x\to -3.5} f'(x)$ equals the slope of the tangent line at $x = -3.5$. Using the points $(-4, 4)$ and $(-3, 8)$ to compute this slope, we get

$$\frac{8-4}{1} = 4 \ .$$