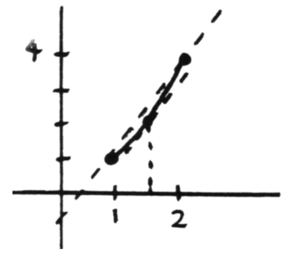
SECTION 4.9 The Mean Value Theorem

IN-SECTION EXERCISES:

EXERCISE 1.

1. For $[a,b]=[1,2]:$

$$\frac{f(b)-f(a)}{b-a}=\frac{f(2)-f(1)}{2-1}=\frac{4-1}{1}=3$$

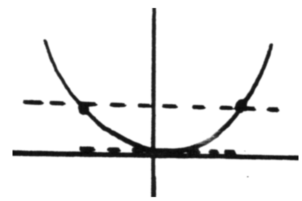


We seek c for which $f'(c)=3$; note that $f'(x)=2x$, so that $f'(c)=2c$. Then:

$$\begin{aligned} f'(c)=3 \quad &\iff \quad 2c=3 \\ &\iff \quad c=\frac{3}{2} \end{aligned}$$

2. For $[a,b]=[-1,1]:$

$$\frac{f(b)-f(a)}{b-a}=\frac{f(1)-f(-1)}{1-(-1)}=\frac{1-1}{2}=0$$

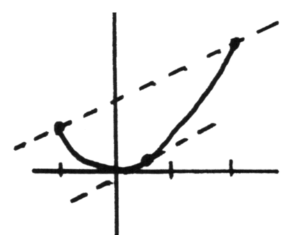


Then:

$$\begin{aligned} f'(c)=0 \quad &\iff \quad 2c=0 \\ &\iff \quad c=0 \end{aligned}$$

3. For $[a,b]=[-1,2]:$

$$\frac{f(b)-f(a)}{b-a}=\frac{f(2)-f(-1)}{2-(-1)}=\frac{4-1}{3}=1$$

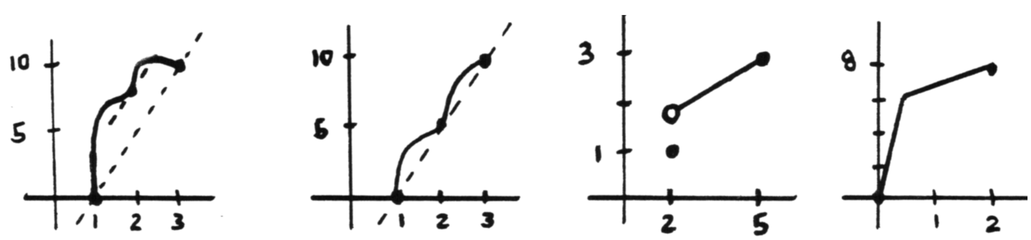


Then:

$$\begin{aligned} f'(c)=1 \quad &\iff \quad 2c=1 \\ &\iff \quad c=\frac{1}{2} \end{aligned}$$

EXERCISE 2.

There are many possible correct graphs.



1. Observe that $\frac{f(3)-f(1)}{3-1}=\frac{10-0}{2}=5$. Thus, the slope of the line through the endpoints must agree with the slope of the tangent line at the point with $x=2$. See the graph above.
2. The point $(2,5)$ must lie on the graph of f; the slope of the tangent line at this point equals the slope of the line through the endpoints.
3. Note that $\frac{f(5)-f(2)}{5-2}=\frac{3-1}{3}=\frac{2}{3}$. IF f were continuous on $[2,5]$, then there would have to be a number $c\in(2,5)$ for which $f'(c)=\frac{2}{3}$. Thus, f must be discontinuous at an endpoint.
4. It must be that f does NOT meet the hypotheses of the Mean Value Theorem.

EXERCISE 3.

By the Mean Value Theorem, for every interval $[a,b]$, there exists $c \in (a,b)$ with:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Since $|f'(x)| \le 2$ for all $x \in \mathbb{R}$, it must be that:

$$\left|\frac{f(b)-f(a)}{b-a}\right| \le 2$$

That is:

$$|f(b)-f(a)| \le 2|b-a|$$

1. Let $[a,b] = [1,2]$, and $f(1) = 5$. Then, $|f(2)-f(1)| = |f(2)-5| \le 2|2-1|$. Thus, the distance from $f(2)$ to 5 must be less than or equal to 2. That is, $f(2) \in (5-2, 5+2)$.
2. Let $[a,b] = [1,3]$, and $f(1) = 5$. Then, $|f(3)-f(1)| = |f(3)-5| \le 2|3-1|$; that is, $|f(3)-5| \le 4$. Thus, $f(3)$ must lie within 4 units of 5. That is, $f(3) \in (5-4, 5+4)$.
3. When x changes by Δx, $f(x)$ can change (at most) by $2\Delta x$. (It could *increase* by $2\Delta x$, or *decrease* by $2\Delta x$.)

EXERCISE 4.

1. TRUE. A number's distance from zero is always greater than or equal to zero.
2. FALSE. Take $x = 0$. Then, the sentence '$|0| > 0$' is false.
3. TRUE. Here, the dummy variable t was used, instead of x, to denote a typical element from the universal set.
4. TRUE. No matter what number x is chosen from the interval $(2,3)$, x is greater than or equal to 0.
5. TRUE. This is the *definition* of the set $A \cap B$.
6. TRUE. This is a precise statement that the derivative of a sum of differentiable functions is the sum of the derivatives.

EXERCISE 5.

1. For all x and y, $x+y = y+x$

   Or, more briefly: $x+y = y+x$
2. For all x, $x = 2 \iff 3x = 6$

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3. For all A and B, $A \subset A \cup B$

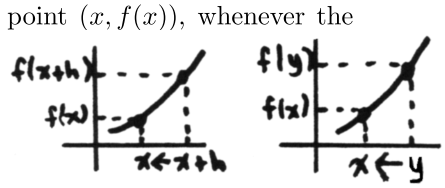
   Or, more briefly: $A \subset A \cup B$
4. For all P and Q, $\big((P \Rightarrow Q) \iff (\text{not } Q \Rightarrow \text{not } P)\big)$. Observe that this is a statement that an implication is equivalent to its contrapositive.

   More briefly: $(P \Rightarrow Q) \iff (\text{not } Q \Rightarrow \text{not } P)$

END-OF-SECTION EXERCISES:

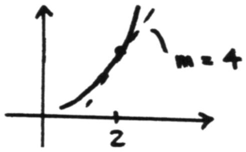
1. The limit gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, whenever the tangent line exists and is non-vertical.
2. Same as (1). Indeed:

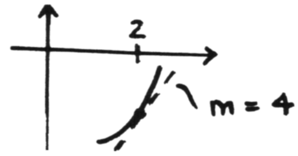
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{y\to x} \frac{f(y)-f(x)}{y-x}$$



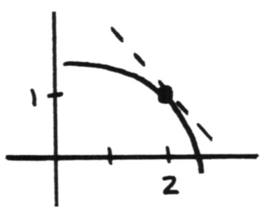
The difference quotients $\frac{f(x+h)-f(x)}{h}$ and $\frac{f(y)-f(x)}{y-x}$ both represent the slope of the secant line through the point $(x, f(x))$ and a nearby point. The nearby point is called $(x+h, f(x+h))$ in the first difference quotient; and is called $(y, f(y))$ (where y is close to x) in the second difference quotient.

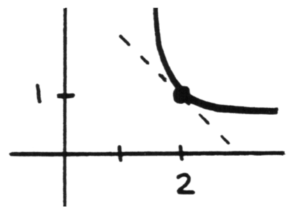
3. There is a tangent line to the graph of f when $x = 2$, and its slope is 4.





4. The graph of f contains the point $(2, 1)$, and the slope of the tangent line at this point is -1.





5. Let $f(x) = -x^2$. Then:

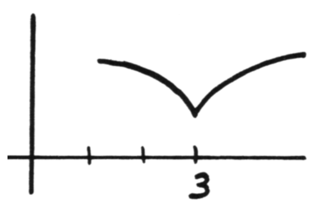
$$\begin{aligned} f'(x) &:= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{-(x+h)^2 - (-x^2)}{h} \\ &= \lim_{h\to 0} \frac{-(x^2 + 2xh + h^2) + x^2}{h} \\ &= \lim_{h\to 0} \frac{h(-2x - h)}{h} \\ &= \lim_{h\to 0} (-2x - h) = -2x \end{aligned}$$

(Using the Simple Power Rule is certainly easier: $f'(x) = -2x$!)

6. Let $f'(x) = 3x$. Then:

$$\begin{aligned} f''(x) &:= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h\to 0} \frac{3(x+h) - 3x}{h} \\ &= \lim_{h\to 0} \frac{3h}{h} = 3 \end{aligned}$$

7. Put a 'kink' in the graph when $x = 3$.



8. If f is differentiable at c, then f is continuous at c. Therefore, it is impossible to find a function that is differentiable at 3, but not continuous at 3.

9. Using a 'generalized' product rule:

$$\begin{aligned}
f'(x) &= (1)e^{2x}\ln(2-x) + x(2e^{2x})\ln(2-x) + xe^{2x}\frac{1}{2-x}(-1)\\
&= e^{2x}\ln(2-x) + 2xe^{2x}\ln(2-x) - \frac{xe^{2x}}{2-x}\\
\mathcal{D}(f) &= \{x \mid 2-x>0\} = \{x \mid -x > -2\} = \{x \mid x<2\} = (-\infty, 2)\\
\mathcal{D}(f') &= \{x \mid x\in\mathcal{D}(f) \text{ and } 2-x>0 \text{ and } 2-x\neq 0\} = (-\infty,2)
\end{aligned}$$

When $x=0$, $f(0)=0$, so the point $(0,0)$ lies on the graph of f. The slope of the tangent line at this point is:

$$f'(0) = e^0\ln(2-0) + 0 - 0 = \ln 2$$

The tangent line has equation $y - 0 = (\ln 2)(x-0)$, that is, $y = (\ln 2)x$.

10. Without using the Chain Rule:

$$\begin{aligned}
f(g(x)) &= f(\frac{1}{x}) = (\frac{1}{x})^2 = \frac{1}{x^2} = x^{-2}\\
\frac{d}{dx}f(g(x)) &= -2x^{-3} = -\frac{2}{x^3}
\end{aligned}$$

Using the Chain Rule:

$$f'(x) = 2x\ , \quad g(x) = x^{-1}\ , \quad g'(x) = -x^{-2} = -\frac{1}{x^2}$$

$$\begin{aligned}
\frac{d}{dx}f(g(x)) &= f'(g(x))\cdot g'(x)\\
&= f'(\frac{1}{x})\cdot(-\frac{1}{x^2})\\
&= 2(\frac{1}{x})\cdot(-\frac{1}{x^2})\\
&= \frac{-2}{x^3}
\end{aligned}$$

Same results (of course)!