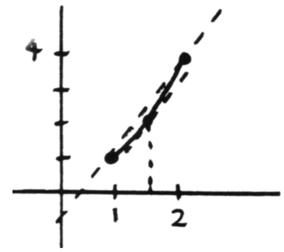
## SECTION 4.9 The Mean Value Theorem

IN-SECTION EXERCISES:

EXERCISE 1.

1. For $[a,b] = [1,2]$,

$$\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(1)}{2-1} = \frac{4-1}{1} = 3$$

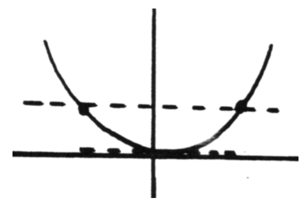



We seek $c$ for which $f'(c) = 3$; note that $f'(x) = 2x$, so that $f'(c) = 2c$. Then:

$$\begin{aligned} f'(c) = 3 \quad &\iff \quad 2c = 3 \\ &\iff \quad c = \frac{3}{2} \end{aligned}$$

2. For $[a,b] = [-1,1]$,

$$\frac{f(b)-f(a)}{b-a} = \frac{f(1)-f(-1)}{1-(-1)} = \frac{1-1}{2} = 0 \ .$$

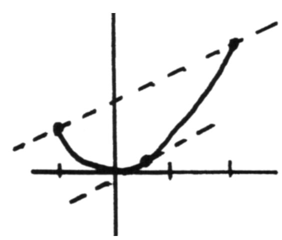


Then:

$$\begin{aligned} f'(c) = 0 \quad &\iff \quad 2c = 0 \\ &\iff \quad c = 0 \end{aligned}$$

3. For $[a,b] = [-1,2]$,

$$\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(-1)}{2-(-1)} = \frac{4-1}{3} = 1 \ .$$

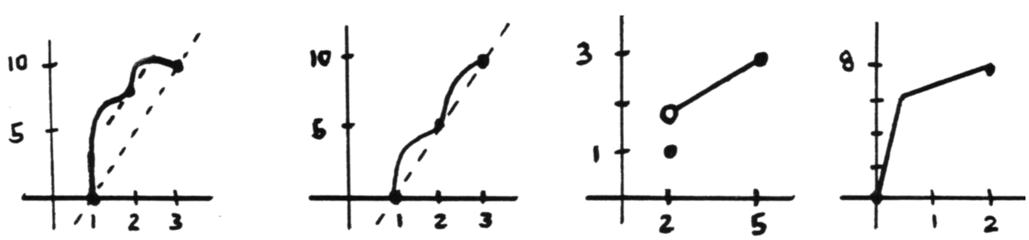


Then:

$$\begin{aligned} f'(c) = 1 \quad &\iff \quad 2c = 1 \\ &\iff \quad c = \frac{1}{2} \end{aligned}$$

EXERCISE 2.

There are many possible correct graphs.




1. Observe that $\frac{f(3)-f(1)}{3-1} = \frac{10-0}{2} = 5$. Thus, the slope of the line through the endpoints must agree with the slope of the tangent line at the point with $x = 2$. See the graph above.
2. The point $(2,5)$ must lie on the graph of $f$; the slope of the tangent line at this point equals the slope of the line through the endpoints.
3. Note that $\frac{f(5)-f(2)}{5-2} = \frac{3-1}{3} = \frac{2}{3}$. IF $f$ were continuous on $[2,5]$, then there would have to be a number $c \in (2,5)$ for which $f'(c) = \frac{2}{3}$. Thus, $f$ must be discontinuous at an endpoint.
4. It must be that $f$ does NOT meet the hypotheses of the Mean Value Theorem.

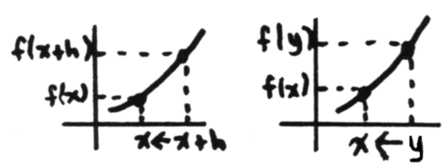
EXERCISE 3.
By the Mean Value Theorem, for every interval $[a,b]$, there exists $c \in (a,b)$ with
$$f'(c) = \frac{f(b)-f(a)}{b-a} \ .$$
Since $|f'(x)| \le 2$ for all $x \in \mathbb{R}$, it must be that
$$\left|\frac{f(b)-f(a)}{b-a}\right| \le 2 \ ;$$
that is,
$$|f(b)-f(a)| \le 2|b-a| \ .$$

1. Let $[a,b]=[1,2]$, and $f(1)=5$. Then, $|f(2)-f(1)| = |f(2)-5| \le 2|2-1|$. Thus, the distance from $f(2)$ to 5 must be less than or equal to 2. That is, $f(2) \in (5-2, 5+2)$.
2. Let $[a,b]=[1,3]$, and $f(1)=5$. Then, $|f(3)-f(1)| = |f(3)-5| \le 2|3-1|$; that is, $|f(3)-5| \le 4$. Thus, $f(3)$ must lie within 4 units of 5. That is, $f(3) \in (5-4, 5+4)$.
3. When $x$ changes by $\Delta x$, $f(x)$ can change (at most) by $2\Delta x$. (It could *increase* by $2\Delta x$, or *decrease* by $2\Delta x$.)

EXERCISE 4.
1. TRUE. A number's distance from zero is always greater than or equal to zero.
2. FALSE. Take $x=0$. Then, the sentence '$|0|>0$' is false.
3. TRUE. Here, the dummy variable $t$ was used, instead of $x$, to denote a typical element from the universal set.
4. TRUE. No matter what number $x$ is chosen from the interval $(2,3)$, $x$ is greater than or equal to 0.
5. TRUE. This is the *definition* of the set $A \cap B$.
6. TRUE. This is a precise statement that the derivative of a sum of differentiable functions is the sum of the derivatives.

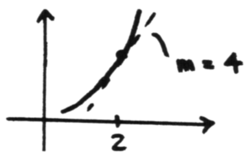
EXERCISE 5.
1. For all $x$ and $y$, $x+y=y+x$.
   Or, more briefly: $x+y=y+x$ .
2. For all $x$, $x=2 \iff 3x=6$.
   Or, more briefly: $x=2 \iff 3x=6$.
3. For all $A$ and $B$, $A \subset A \cup B$.
   Or, more briefly: $A \subset A \cup B$.
4. For all $P$ and $Q$, $\big((P \Rightarrow Q) \iff (\text{not } Q \Rightarrow \text{not } P)\big)$. Observe that this is a statement that an implication is equivalent to its contrapositive.
   More briefly: $(P \Rightarrow Q) \iff (\text{not } Q \Rightarrow \text{not } P)$.

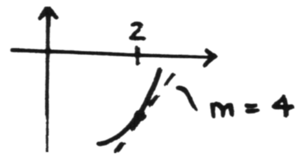
END-OF-SECTION EXERCISES:
1. The limit gives the slope of the tangent line to the graph of $f$ at the point $(x, f(x))$, whenever the tangent line exists and is non-vertical.
2. Same as (1). Indeed:
$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{y\to x}\frac{f(y)-f(x)}{y-x}$$



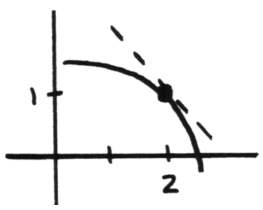

   The difference quotients $\frac{f(x+h)-f(x)}{h}$ and $\frac{f(y)-f(x)}{y-x}$ both represent the slope of the secant line through the point $(x, f(x))$ and a nearby point. The nearby point is called $(x+h, f(x+h))$ in the first difference quotient; and is called $(y, f(y))$ (where $y$ is close to $x$) in the second difference quotient.

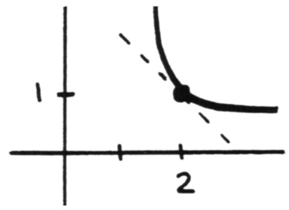
3. There is a tangent line to the graph of $f$ when $x = 2$, and its slope is 4.




4. The graph of $f$ contains the point $(2, 1)$, and the slope of the tangent line at this point is $-1$.




5. Let $f(x) = -x^2$. Then:

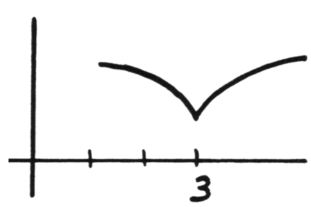
$$\begin{aligned} f'(x) &:= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{-(x+h)^2 - (-x^2)}{h} \\ &= \lim_{h\to 0} \frac{-(x^2 + 2xh + h^2) + x^2}{h} \\ &= \lim_{h\to 0} \frac{h(-2x - h)}{h} \\ &= \lim_{h\to 0} (-2x - h) = -2x \end{aligned}$$

(Using the Simple Power Rule is certainly easier: $f'(x) = -2x$!)

6. Let $f'(x) = 3x$. Then:

$$\begin{aligned} f''(x) &:= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h\to 0} \frac{3(x+h) - 3x}{h} \\ &= \lim_{h\to 0} \frac{3h}{h} = 3 \end{aligned}$$

7. Put a 'kink' in the graph when $x = 3$.




8. If $f$ is differentiable at $c$, then $f$ is continuous at $c$. Therefore, it is impossible to find a function that is differentiable at 3, but not continuous at 3.

9. Using a 'generalized' product rule:

$$\begin{aligned}
f'(x) &= (1)e^{2x}\ln(2-x) + x(2e^{2x})\ln(2-x) + xe^{2x}\frac{1}{2-x}(-1) \\
&= e^{2x}\ln(2-x) + 2xe^{2x}\ln(2-x) - \frac{xe^{2x}}{2-x} \\
\mathcal{D}(f) &= \{x \mid 2-x>0\} = \{x \mid -x>-2\} = \{x \mid x<2\} = (-\infty,2) \\
\mathcal{D}(f') &= \{x \mid x\in\mathcal{D}(f) \text{ and } 2-x>0 \text{ and } 2-x\neq 0\} = (-\infty,2)
\end{aligned}$$

When $x=0$, $f(0)=0$, so the point $(0,0)$ lies on the graph of $f$. The slope of the tangent line at this point is:

$$f'(0) = e^0\ln(2-0) + 0 - 0 = \ln 2 \ ;$$

the tangent line has equation $y-0=\ln 2(x-0)$ , that is, $y=(\ln 2)x$.

10. Without using the Chain Rule:

$$\begin{aligned}
f(g(x)) &= f(\frac{1}{x}) = (\frac{1}{x})^2 = \frac{1}{x^2} = x^{-2}\ , \\
\frac{d}{dx}f(g(x)) &= -2x^{-3} = -\frac{2}{x^3}
\end{aligned}$$

Using the Chain Rule:

$$f'(x) = 2x\ ,\quad g(x)=x^{-1}\ ,\quad g'(x) = -x^{-2} = -\frac{1}{x^2}\ ,$$

$$\begin{aligned}
\frac{d}{dx}f(g(x)) &= f'(g(x))\cdot g'(x) \\
&= f'(\frac{1}{x})\cdot(-\frac{1}{x^2}) \\
&= 2(\frac{1}{x})\cdot(-\frac{1}{x^2}) \\
&= \frac{-2}{x^3}
\end{aligned}$$

Same results (of course)!