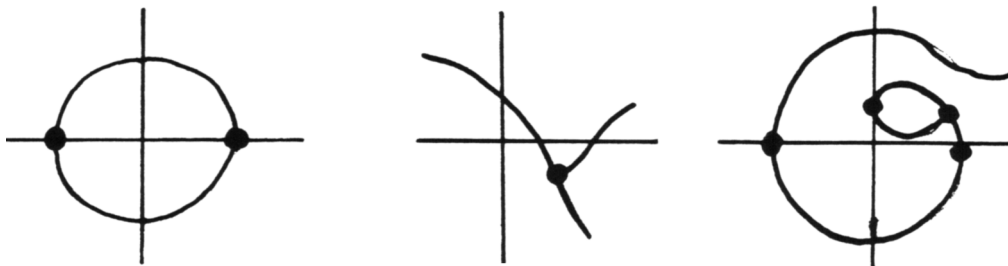


SECTION 4.8 Implicit Differentiation (Optional)

IN-SECTION EXERCISES:

EXERCISE 1.

The points where y is NOT locally a function of x are identified in the graphs below.



EXERCISE 2.

The final results are given in both *prime* and *Leibniz* notation.

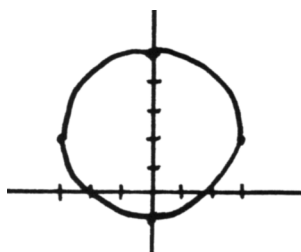
1. $\frac{d}{dx}(y^2) = 2y \frac{dy}{dx} = 2yy'$
2. $\frac{d}{dx}(xy) = x \frac{dy}{dx} + (1)y = xy' + y$
3. $\frac{d}{dx}(x+y)^3 = 3(x+y)^2(1 + \frac{dy}{dx}) = 3(x+y)^2(1+y')$
4. $\frac{d}{dx}(\ln y) = \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{y} \cdot y'$

EXERCISE 3.

$$\begin{aligned}
 y &= -(1-x^2)^{1/2}, \\
 \frac{dy}{dx} &= -\frac{1}{2}(1-x^2)^{-1/2}(-2x) \\
 &= \frac{x}{\sqrt{1-x^2}} \\
 &= -\frac{x}{-\sqrt{1-x^2}} \\
 &= -\frac{x}{y}
 \end{aligned}$$

EXERCISE 4.

1. The graph of $x^2 + (y-2)^2 = 3^2$ is the circle centered at $(0, 2)$ with radius 3.



2. y is NOT locally a function of x at the points where there are vertical tangent lines: $(3, 2)$ and $(-3, 2)$.

3. Differentiating implicitly, $2(y-2)^1 \frac{dy}{dx} + 2x = 0$. Solving for $\frac{dy}{dx}$:

$$\begin{aligned} 2(y-2)^1 \frac{dy}{dx} + 2x = 0 &\iff 2(y-2) \frac{dy}{dx} = -2x \\ &\iff \frac{dy}{dx} = \frac{-2x}{2(y-2)} \\ &\iff \frac{dy}{dx} = -\frac{x}{y-2} = \frac{x}{2-y} \end{aligned}$$

Observe that the formula fails when $y = 2$; these are precisely the points where y is NOT locally a function of x .

EXERCISE 5.

1. First, find the natural logarithm of y :

$$\begin{aligned} \ln y &= \ln(x^{-1}(2x-1)^{-1}(3x-1)^{-1}) \\ &= \ln x^{-1} + \ln(2x-1)^{-1} + \ln(3x-1)^{-1} \\ &= -\ln x - \ln(2x-1) - \ln(3x-1) \\ &= -(\ln x + \ln(2x-1) + \ln(3x-1)); \end{aligned}$$

Then, differentiate implicitly:

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= (-1) \left(\frac{1}{x} + \frac{1}{2x-1}(2) + \frac{1}{3x-1}(3) \right) \\ &= - \left(\frac{1}{x} + \frac{2}{2x-1} + \frac{3}{3x-1} \right) \end{aligned}$$

Solving for $\frac{dy}{dx}$ yields:

$$\begin{aligned} \frac{dy}{dx} &= -y \left(\frac{1}{x} + \frac{2}{2x-1} + \frac{3}{3x-1} \right) \\ &= -\frac{1}{x(2x-1)(3x-1)} \left(\frac{1}{x} + \frac{2}{2x-1} + \frac{3}{3x-1} \right) \end{aligned}$$

2. You should be able to take the natural logarithm of y and simplify, all in one step:

$$\ln y = 4 \ln x + \frac{1}{3} \ln(x-1) - \frac{1}{5} \ln(2x+1)$$

Then, implicit differentiation yields

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{4}{x} + \frac{1}{3(x-1)} - \frac{2}{5(2x+1)}$$

so that:

$$\frac{dy}{dx} = \frac{x^4 \sqrt[3]{x-1}}{\sqrt[5]{2x+1}} \cdot \left[\frac{4}{x} + \frac{1}{3(x-1)} - \frac{2}{5(2x+1)} \right]$$

EXERCISE 6.

1.

$$\begin{aligned}\ln y &= x \ln x , \\ \frac{1}{y} \cdot \frac{dy}{dx} &= x \cdot \frac{1}{x} + (1) \ln x , \\ \frac{dy}{dx} &= x^x(1 + \ln x)\end{aligned}$$

2.

$$\begin{aligned}\ln y &= x \ln(2x) , \\ \frac{1}{y} \cdot \frac{dy}{dx} &= x \cdot \frac{1}{2x} \cdot 2 + (1) \ln(2x) , \\ \frac{dy}{dx} &= (2x)^x[1 + \ln(2x)]\end{aligned}$$

3.

$$\begin{aligned}\ln y &= 3x \ln(2x) , \\ \frac{1}{y} \cdot \frac{dy}{dx} &= 3x \cdot \frac{1}{2x} \cdot 2 + (3) \ln(2x) , \\ \frac{dy}{dx} &= (2x)^{3x}[3 + 3 \ln(2x)]\end{aligned}$$

4.

$$\begin{aligned}y &= [(x+1)^{1/2}]^{(x^2)} \\ &= (x+1)^{(\frac{1}{2}x^2)} , \\ \ln y &= \frac{1}{2}x^2 \ln(x+1) , \\ \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{1}{2}x^2 \frac{1}{x+1} + (x) \ln(x+1) , \\ \frac{dy}{dx} &= (\sqrt{x+1})^{(x^2)} \left[\frac{x^2}{2(x+1)} + x \ln(x+1) \right]\end{aligned}$$

EXERCISE 7.

- Let $a = 2$ and $b = -2$. Then the sentence ' $a = b$ ' becomes ' $2 = -2$ ', which is false. But the sentence ' $a^2 = b^2$ ' becomes ' $2^2 = (-2)^2$ ', which is true.
- To show that a sentence of the form ' $A \iff B$ ' is true, one can show that both ' $A \implies B$ ' and ' $B \implies A$ ' are true. This is justified by the logical equivalence

$$A \iff B \iff [(A \implies B) \text{ and } (B \implies A)] ;$$

see the truth table below.

A	B	$A \iff B$	$A \implies B$	$B \implies A$	$A \implies B$ AND $B \implies A$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Proof. Let $a \geq 0$ and $b \geq 0$.

‘ \implies ’,

Suppose that $a = b$. Then, $a^2 = b^2$. (This direction is true for *all* real numbers a and b .)

‘ \impliedby ’,

Suppose that $a^2 = b^2$. Then, taking square roots (correctly!) yields $|a| = |b|$. Since a and b are nonnegative, $|a| = a$ and $|b| = b$. Thus, $a = b$.

3. Thus, whenever a and b are both known to be nonnegative, the sentence ‘ $a = b$ ’ always has the same truth value as the sentence ‘ $a^2 = b^2$ ’, so these two sentences can be used interchangeably.

END-OF-SECTION EXERCISES:

1. Completing the square:

$$\begin{aligned} x^2 + 4x + y^2 - 2y + 4 = 0 &\iff (x^2 + 4x + (\frac{4}{2})^2) + (y^2 - 2y + (\frac{-2}{2})^2) = -4 + 4 + 1 \\ &\iff (x + 2)^2 + (y - 1)^2 = 1 ; \end{aligned}$$

the graph is the circle centered at $(-2, 1)$ with radius 1.

y is NOT locally a function of x at the points $(-1, 1)$ and $(-3, 1)$. (There are vertical tangent lines here.)

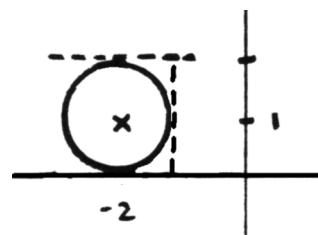
Differentiating implicitly:

$$2(x + 2) + 2(y - 1)y' = 0 ;$$

solving for y' yields

$$y' = \frac{-2(x + 2)}{2(y - 1)} = \frac{-(x + 2)}{y - 1}$$

Note that the formula fails when $y = 1$.



The point $(-2, 2)$ lies on the circle since

$$(-2 + 2)^2 + (2 - 1)^2 = 0^2 + 1^2 = 1 .$$

At this point, the tangent line has slope

$$y'|_{(-2,2)} = \frac{-(-2 + 2)}{2 - 1} = 0 ,$$

so there is a horizontal tangent line (as expected). The equation of the tangent line at the point $(-2, 2)$ is $y = 2$.

The point $(-1, 1)$ lies on the circle since

$$(-1 + 2)^2 + (1 - 1)^2 = 1^2 + 0^2 = 1 .$$

At this point, the formula for y' fails; the tangent line is vertical, and has equation $x = -1$.

2. Same circle as (1), hence the same derivative. At $(-2, 0)$ there is a horizontal tangent line, with equation $y = 0$. At $(-3, 1)$ there is a vertical tangent line, with equation $x = -3$.

3. Put the circle in standard form, by completing the square:

$$\begin{aligned} 4x - 2y = -x^2 - y^2 - 1 &\iff x^2 + 4x + y^2 - 2y = -1 \\ &\iff (x^2 + 4x + 4) + (y^2 - 2y + 1) = -1 + 4 + 1 \\ &\iff (x + 2)^2 + (y - 1)^2 = 2^2 ; \end{aligned}$$

the graph is the circle centered at $(-2, 1)$ with radius 2.

y is NOT locally a function of x at the points $(0, 1)$ and $(-4, 1)$; there are vertical tangent lines here.

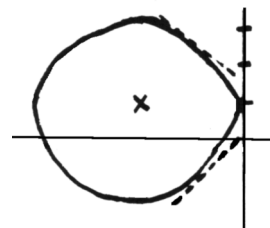
Differentiating implicitly,

$$2(x + 2) + 2(y - 1)y' = 0 ;$$

solving for y' yields

$$y' = -\frac{2(x + 2)}{2(y - 1)} = -\frac{x + 2}{y - 1} .$$

Note that the formula fails when $y = 1$.



The point $(-1, 1 + \sqrt{3})$ lies on the circle since

$$(-1 + 2)^2 + (1 + \sqrt{3} - 1)^2 = 1 + (\sqrt{3})^2 = 1 + 3 = 4 ;$$

the slope of the tangent line at this point is

$$y'|_{(-1, 1 + \sqrt{3})} = -\frac{-1 + 2}{1 + \sqrt{3} - 1} = -\frac{1}{\sqrt{3}} .$$

The equation of the tangent line is

$$y - (1 + \sqrt{3}) = -\frac{1}{\sqrt{3}}(x - (-1)) .$$

4. Same circle as (3); hence the same derivative. The point $(-1, 1 - \sqrt{3})$ lies on the graph since

$$(-1 + 2)^2 + (1 - \sqrt{3} - 1)^2 = 1 + (-\sqrt{3})^2 = 1 + 3 = 4 ;$$

the slope of the tangent line at this point is

$$y'|_{(-1, 1 - \sqrt{3})} = -\frac{-1 + 2}{1 - \sqrt{3} - 1} = -\frac{1}{-\sqrt{3}} = \frac{1}{\sqrt{3}} .$$

The equation of the tangent line is

$$y - (1 - \sqrt{3}) = \frac{1}{\sqrt{3}}(x - (-1)) .$$