SECTION 4.7 Higher Order Derivatives

IN-SECTION EXERCISES:

EXERCISE 1.

1. g''
2. $g''(x)$
3. $(f''')' = f^{(4)}$
4. $f^{(8)}(3)$

EXERCISE 2.

$$\begin{aligned} P'(x) &= 14x^6 - 3x^2 \\ P''(x) &= 84x^5 - 6x \\ P'''(x) &= 420x^4 - 6 \\ P^{(4)}(x) &= 1680x^3 \\ P^{(5)}(x) &= 5040x^2 \\ P^{(6)}(x) &= 10080x \\ P^{(7)}(x) &= 10080 \\ P^{(n)}(x) &= 0\ , \quad \text{for } n \ge 8 \end{aligned}$$

EXERCISE 3.

1.

$$\sum_{j=1}^{6} b_j = b_1 + b_2 + b_3 + b_4 + b_5 + b_6$$

$$\sum_{k=1}^{5} (k+1)^k = (1+1)^1 + (2+1)^2 + (3+1)^3 + (4+1)^4 + (5+1)^5$$

$$\sum_{m=0}^{4} (m+1) = (0+1) + (1+1) + (2+1) + (3+1) + (4+1)$$

$$\sum_{i=1}^{n} 2i = 2(1) + 2(2) + \cdots + 2(n-1) + 2n$$

2. There are many possibilities: $\displaystyle\sum_{i=1}^{n} 2i = \sum_{j=1}^{n} 2j = \sum_{m=1}^{n} 2m = \sum_{k=1}^{n} 2k$

   The variable n CANNOT be used as a dummy variable in this problem. Why not?

3.

$$\begin{aligned} \sum_{j=1}^{n} ka_j &= ka_1 + ka_2 + \cdots + ka_n \\ &= k(a_1 + a_2 + \cdots + a_n) \\ &= k \sum_{j=1}^{n} a_j \end{aligned}$$

4. Here are some correct answers. (Other correct answers are also possible.)

$$
\begin{aligned}
1 + 2 + 3 + \cdots + 100 &= \sum_{i=1}^{100} i \\
34 + 35 + 36 + \cdots + 79 &= \sum_{i=34}^{79} i \\
2 + 4 + 6 + \cdots + 78 &= \sum_{i=1}^{39} 2i \\
5^2 + 6^3 + 7^4 + 8^5 + \cdots + 20^{17} &= \sum_{i=2}^{17} (i+3)^i
\end{aligned}
$$

5. The statement merely states (in summation notation) that the derivative of a sum is the sum of the derivatives.

$$
\begin{aligned}
\frac{d}{dx}\sum_{i=1}^{n} f_i(x) &= \frac{d}{dx}(f_1(x) + \cdots + f_n(x)) \\
&= f_1'(x) + \cdots + f_n'(x) \\
&= \sum_{i=1}^{n} f_i'(x)
\end{aligned}
$$

EXERCISE 4.

1.

$$
\begin{aligned}
P(x) &= a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 \\
&= a_0 + a_1x + a_2x^2 + a_3x^3
\end{aligned}
$$

There are 4 terms in the sum.

2.

$$
\begin{aligned}
\sum_{i=1}^{3} i \cdot a_ix^{i-1} &= 1 \cdot a_1x^{1-1} + 2 \cdot a_2x^{2-1} + 3 \cdot a_3x^{3-1} \\
&= a_1 + 2a_2x + 3a_3x^2
\end{aligned}
$$

Compare with:

$$
\frac{d}{dx}(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2
$$

3.

$$
\begin{aligned}
P''(x) &= \sum_{i=2}^{3} i(i-1)a_ix^{i-2} \\
P'''(x) &= \sum_{i=3}^{3} i(i-1)(i-2)a_ix^{i-3}
\end{aligned}
$$

The formula for $P'''(x)$ collapses to a single term:

$$
\sum_{i=3}^{3} i(i-1)(i-2)a_ix^{i-3} = 3(2)(1)a_3x^{3-3} = 6a_3
$$

4. For $n \ge 4$, $P^{(n)}(x) = 0$

EXERCISE 5.

1.
$$\begin{aligned} 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ 0! &= 1 \\ 100! &= 100 \cdot 99 \cdot \ldots \cdot 2 \cdot 1 \end{aligned}$$

2.
$$\begin{aligned} 10 \cdot 9 \cdot 8 \ldots \cdot 2 \cdot 1 &= 10! \\ 207 \cdot 206 \cdot 205 \cdot \ldots \cdot 1 &= 207! \end{aligned}$$

3.
$$\begin{aligned} 105 \cdot 104 \cdot 103 \cdot \ldots \cdot 50 &= 105 \cdot \ldots \cdot 50 \cdot \frac{49!}{49!} \\ &= \frac{105!}{49!} \end{aligned}$$

EXERCISE 6.

1. $\frac{d^2y}{dx^2}$
2. $\frac{d^2y}{dt^2}$
3. $\frac{d^2g}{dx^2}$
4. $\frac{d^2g}{dx^2}|_{x=2}$ or $\frac{d^2g}{dx^2}(2)$
5. $\frac{d^4y}{dx^4}$
6. $\frac{d^5y}{dx^5}(3)$

EXERCISE 7.

1.
$$\begin{aligned} y &= xe^{-x} \\ y' &= x(-e^{-x}) + (1)e^{-x} = -xe^{-x} + e^{-x} \\ y'' &= (-x)(-e^{-x}) + (-1)e^{-x} + (-e^{-x}) \\ &= xe^{-x} - 2e^{-x} \end{aligned}$$

2.
$$\begin{aligned} f(x) &= (x-1)^{-1} + (x-2)^{-1} \\ f'(x) &= -(x-1)^{-2} - (x-2)^{-2} \\ f''(x) &= 2(x-1)^{-3} + 2(x-2)^{-3} \\ &= \frac{2}{(x-1)^3} + \frac{2}{(x-2)^3} \end{aligned}$$

3. The results of problem (1) can be used; it was found that $f'(x) = -xe^{-x} + e^{-x}$. When $x = 0$, $f'(0) = 0 + 1 = 1$, so the point $(0, 1)$ lies on the graph of f'. The slope of the tangent line at this point is $f''(0) = (xe^{-x} - 2e^{-x})|_{x=0} = -2$. The equation of the desired tangent line is $y - 1 = -2(x - 0)$, or equivalently, $y = -2x + 1$.

END-OF-SECTION EXERCISES:

1. SEN; TRUE
2. SEN; TRUE

3. EXP
4. EXP
5. SEN; CONDITIONAL. This is true if f is of the form $f(x) = x^2 + K$, where $K \in \mathbb{R}$, and is false otherwise.
6. SEN; CONDITIONAL. This is true if y is of the form $y = 3x + K$, where $K \in \mathbb{R}$, and false otherwise. (The expression for y can be written using other dummy variables, e.g., $y = 3t + K$.)
7. SEN; TRUE. This sentence states that the derivative of a sum is the sum of the derivatives.
8. SEN; TRUE. $f'(c)$ is the prime notation for f', evaluated at c; $\frac{df}{dx}(c)$ is the Leibniz notation for the same.
9. EXP
10. SEN; TRUE. The log of a product is the sum of the logs.
11. EXP
12. SEN. As long as f is differentiable at $g(x)$ and g is differentiable at x, then this sentence is true. This is a statement of the Chain Rule, the rule which tells how to differentiate composite functions.
13. EXP
14. SEN; TRUE
15. SEN; TRUE, since $\sum_{i=0}^{3} i = 0 + 1 + 2 + 3 = 6$
16. EXP
17. SEN; CONDITIONAL. This is true only if the slope of the tangent line at the point $(c, f(c))$ equals 2.
18. SEN; FALSE. Remember that 'if and only if' is another way to say 'is equivalent to'. For two mathematical sentences to be equivalent, they must *always have the same truth value.* However, there are functions for which the statement 'f is continuous at c' is true, BUT the statement 'f is differentiable at c' is false. Just 'kink' the graph of f at c!