SECTION 4.5 The Chain Rule (Differentiating Composite Functions)

IN-SECTION EXERCISES:

EXERCISE 1.

1.
$$\begin{aligned}(f \circ g)(x) &:= f(g(x)) \\ &= f(-2x) = (-2x)^2 - 1 \\ &= 4x^2 - 1 \\ (g \circ f)(x) &:= g(f(x)) \\ &= g(x^2 - 1) = -2(x^2 - 1) \\ &= -2x^2 + 2\end{aligned}$$

2. One correct solution is to define $g(x) = 2x$ and $f(x) = (x+1)^2$. Then:

$$h(x) := f(g(x)) = f(2x) = (2x+1)^2$$

Another correct solution is to define $g(x) = 2x + 1$, and $f(x) = x^2$. Then:

$$h(x) = f(g(x)) = f(2x+1) = (2x+1)^2$$

EXERCISE 2.

1. Since $(f \circ g)(x) = x^2 + 2x + 1$, differentiation yields $(f \circ g)'(x) = 2x + 2$. Thus, $(f \circ g)'(2) = 2(2) + 2 = 6$.
2. Since $g(x) = x + 1$, $g'(x) = 1$. Thus, $g'(2) = 1$.
3. Since $f(x) = x^2$, $f'(x) = 2x$. Also, $g(2) = 2 + 1 = 3$. Thus, $f'(g(2)) = f'(3) = 2 \cdot 3 = 6$.
4. Multiplying, $f'(g(2)) \cdot g'(2) = 6 \cdot 1 = 6$. Same answer!

EXERCISE 3.

1. Now, $(g \circ f)(x) := g(f(x)) = g(x^2) = x^2 + 1$. Differentiation yields $(g \circ f)'(x) = 2x$. Thus, $(g \circ f)'(2) = 2(2) = 4$.
2. We must have:
$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$
In particular:
$$(g \circ f)'(2) = g'(f(2)) \cdot f'(2)$$
Since $f(x) = x^2$, $f'(x) = 2x$. Thus, $f'(2) = 2 \cdot 2 = 4$.
3. Since $g(x) = x + 1$, $g'(x) = 1$. Also, $f(2) = 2^2 = 4$. Thus, $g'(f(2)) = g'(4) = 1$. (Observe that we didn't really need to find $f(2)$, since $g'(\text{anything}) = 1$.)
4. Multiplying, $g'(f(2)) \cdot f'(2) = 1 \cdot 4 = 4$. Same answer!

EXERCISE 4.

Method I: First find the function $f \circ g$:

$$\begin{aligned}(f \circ g)(x) = f(g(x)) &= f(3x^2 - 2x) = (3x^2 - 2x)^3 \\ &= (3x^2)^3 + 3(3x^2)^2(-2x) + 3(3x^2)(-2x)^2 + (-2x)^3 \\ &= 27x^6 - 54x^5 + 36x^4 - 8x^3\end{aligned}$$

Pascal's Triangle was used to help expand $(3x^2 - 2x)^3$.

Then, differentiation yields:

$$(f \circ g)'(x) = 162x^5 - 270x^4 + 144x^3 - 24x^2$$

Method II: By the chain rule, $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$.

$$\begin{aligned}
f'(x) &= 3x^2 \\
f'(g(x)) &= f'(3x^2 - 2x) = 3(3x^2 - 2x)^2 \\
g'(x) &= 6x - 2 \\
f'(g(x)) \cdot g'(x) &= 3(3x^2 - 2x)^2(6x - 2) \\
&= 3(9x^4 - 12x^3 + 4x^2)(6x - 2) \\
&= (27x^4 - 36x^3 + 12x^2)(6x - 2) \\
&= 162x^5 - 54x^4 - 216x^4 + 72x^3 + 72x^3 - 24x^2 \\
&= 162x^5 - 270x^4 + 144x^3 - 24x^2
\end{aligned}$$

The answers agree! Normally, there is no advantage to multiplying out the expression $3(3x^2 - 2x)^2(6x - 2)$ that one gets from using the Chain Rule. It was only multiplied out here to compare with the first answer.

EXERCISE 5.

$$(a \circ b \circ c \circ d \circ e)'(x) = a'(b(c(d(e(x))))) \cdot b'(c(d(e(x)))) \cdot c'(d(e(x))) \cdot d'(e(x)) \cdot e'(x)$$

Here:
e must be differentiable at x
d must be differentiable at $e(x)$
c must be differentiable at $d(e(x))$
b must be differentiable at $c(d(e(x)))$
a must be differentiable at $b(c(d(e(x))))$

EXERCISE 6.

1. Method I: Write y as a function of x, and differentiate.

$$\begin{aligned}
y &= 3u = 3(x^2 - 1) = 3x^2 - 3 \\
\frac{dy}{dx} &= 6x
\end{aligned}$$

Method II: Use the Chain Rule.

$$\frac{dy}{du} = 3 \ , \qquad \frac{du}{dx} = 2x$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= 3(2x) = 6x
\end{aligned}$$

2. $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dx}$

3. Method I: Write y as a function of x, and differentiate.

$$y = u^2 = (3v)^2 = 9v^2 = 9(x^3)^2 = 9x^6$$
$$\frac{dy}{dx} = 54x^5$$

Method II: Use the Chain Rule.

$$\frac{dy}{du} = 2u\ , \qquad \frac{du}{dv} = 3\ , \qquad \frac{dv}{dx} = 3x^2$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx} \\ &= 2u\cdot 3\cdot 3x^2 = 18\cdot u\cdot x^2 \\ &= 18\cdot(3v)\cdot x^2 = 54\cdot v\cdot x^2 \\ &= 54\cdot x^3\cdot x^2 = 54x^5\end{aligned}$$

The answers agree!

EXERCISE 7.

Here are some of the things that must be 'worried about':

- g must be differentiable at x (so that $g'(x)$ makes sense)
- both $(g(x))^n$ and $n(g(x))^{n-1}$ must be defined; in particular, one must watch out for division by zero (in the case where $n-1$ is negative)
- watch out for even roots of negative numbers

EXERCISE 8.

1. $f'(x) = 7(2x+1)^6\cdot 2 = 14(2x+1)^6$

   $f'(0) = 14(1)^6 = 14$; $f'(1) = 14(2+1)^6 = 14\cdot 3^6$

2. First, rewrite f: $f(x) = -(x^2+3)^{-1/2}$

   Then:

$$\begin{aligned}f'(x) &= (-1)(-\frac{1}{2})(x^2+3)^{-\frac{1}{2}-1}\cdot(2x) = \frac{1}{2}(x^2+3)^{-\frac{3}{2}}\cdot 2x \\ &= \frac{x}{(x^2+3)^{\frac{3}{2}}} = \frac{x}{\sqrt{(x^2+3)^3}}\end{aligned}$$

   Note that f' was rewritten in a form that closely resembles the original form of f.

   Now, $f'(0) = 0$ (the numerator is 0); and:

$$f'(1) = \frac{1}{\sqrt{(1^2+3)^3}} = \frac{1}{\sqrt{4^3}} = \frac{1}{(\sqrt{4})^3} = \frac{1}{8}$$

3. Note how the $\frac{d}{dx}$ operator is used below:

$$\begin{aligned} f'(x) &= 3(g(h(x)))^2 \frac{d}{dx}(g(h(x))) && \text{(general power rule)} \\ &= 3(g(h(x)))^2 \cdot g'(h(x)) \cdot h'(x) && \text{(chain rule)} \end{aligned}$$

Then, $f'(0) = 3(g(h(0)))^2 \cdot g'(h(0)) \cdot h'(0)$. This expression cannot be simplified further without additional information about the functions g and h. In particular, remember that $h(0)$ is the function h, *evaluated at* 0, (NOT h times 0).

Also: $f'(1) = 3(g(h(1)))^2 \cdot g'(h(1)) \cdot h'(1)$

4. Note how convenient the $\frac{d}{dx}$ operator is for intermediate steps:

$$\begin{aligned} f'(x) &= (-3)[x + (x^2-1)^{-2}]^{-3-1} \cdot \frac{d}{dx}\big(x + (x^2-1)^{-2}\big) && \text{(general power rule)} \\ &= -3[x + (x^2-1)^{-2}]^{-4} \cdot \big(1 + (-2)(x^2-1)^{-2-1} \cdot (2x)\big) && \text{(derivative of sum, general power rule)} \\ &= -3[x + (x^2-1)^{-2}]^{-4} \cdot (1 - 4x(x^2-1)^{-3}) \end{aligned}$$

Then, $f'(0) = -3[(-1)^{-2}]^{-4} = -3$. The functions f and f' are not defined at $x = 1$, since substitution into $(x^2-1)^{-2}$ would cause division by zero.

EXERCISE 9.

Define $f(x) = \ln x$, so that: $(f \circ g)(x) = f(g(x)) = \ln(g(x))$

Recall that $f'(x) = \frac{1}{x}$. Then:

$$\begin{aligned} (f \circ g)'(x) &= f'(g(x)) \cdot g'(x) \\ &= \frac{1}{g(x)} \cdot g'(x) \end{aligned}$$

In order for $\ln(g(x))$ to be defined, $g(x)$ must be positive. Also, g must be differentiable at x, so that $g'(x)$ is defined. (Note: it is also true that $\frac{d}{dx}(\ln|g(x)|) = \frac{1}{g(x)} \cdot g'(x)$, wherever g is differentiable and $g(x) \neq 0$. Try to prove it!)

END-OF-SECTION EXERCISES:

1. $f(x) = 2(e^x - 1)^{-1/2} + x$; then:

$$\begin{aligned} f'(x) &= 2(-\frac{1}{2})(e^x-1)^{-\frac{1}{2}-1}(e^x) + 1 \\ &= -e^x(e^x-1)^{-\frac{3}{2}} + 1 = \frac{-e^x}{\sqrt{(e^x-1)^3}} + 1 \end{aligned}$$

2. $g(x) = \sqrt[3]{x^2-1} = (x^2-1)^{1/3}$; then:

$$\begin{aligned} g'(x) &= \frac{1}{3}(x^2-1)^{\frac{1}{3}-1}(2x) \\ &= \frac{2}{3}x(x^2-1)^{-\frac{2}{3}} = \frac{2x}{3\sqrt[3]{(x^2-1)^2}} \end{aligned}$$

3. $y = (e^x)^3 = e^{3x}$; then: $\frac{dy}{dx} = e^{3x} \cdot 3 = 3e^{3x}$
4. $y' = 3e^{3x}$
5. $y' = 11(3t-4)^{10} \cdot 3 = 33(3t-4)^{10}$
6. $\frac{dy}{dt} = 8(2-t)^7(-1) = -8(2-t)^7$

7. $g(t) = 3\sqrt[6]{t^2+t+1} = 3(t^2+t+1)^{1/6}$; then:

$$\begin{aligned} g'(t) &= 3(\frac{1}{6})(t^2+t+1)^{\frac{1}{6}-1} \cdot (2t+1) \\ &= \frac{1}{2}(t^2+t+1)^{-\frac{5}{6}}(2t+1) \\ &= \frac{2t+1}{2\sqrt[6]{(t^2+2t+1)^5}} \end{aligned}$$

8. $h(t) = -\sqrt[3]{\dfrac{1}{t^2-1}} = -\dfrac{1}{\sqrt[3]{t^2-1}} = -(t^2-1)^{-\frac{1}{3}}$; then:

$$h'(t) = (-1)(-\frac{1}{3})(t^2-1)^{-\frac{1}{3}-1}(2t) = \frac{2t}{3}(t^2-1)^{-\frac{4}{3}} = \frac{2t}{3\sqrt[3]{(t^2-1)^4}}$$

9. $f'(y) = 7e^{-y}(-1) + \frac{1}{-y}(-1) = -7e^{-y} + \frac{1}{y}$

10. $g(y) = \ln\sqrt[3]{-y} = \ln(-y)^{\frac{1}{3}} = \frac{1}{3}\ln(-y)$; then: $g'(y) = \dfrac{1}{3}\dfrac{1}{(-y)}(-1) = \dfrac{1}{3y}$

11. $\dfrac{dy}{dx} = 3(\ln x)^2 \cdot \dfrac{1}{x} = \dfrac{3}{x}(\ln x)^2$

12. $y = \ln(\sqrt{x}(x+1)) = \ln\sqrt{x} + \ln(x+1) = \frac{1}{2}\ln x + \ln(x+1)$; then: $\dfrac{dy}{dx} = \dfrac{1}{2}\cdot\dfrac{1}{x} + \dfrac{1}{x+1}(1) = \dfrac{1}{2x} + \dfrac{1}{x+1}$

13. $y = \dfrac{-1}{t+\sqrt{t-1}} = -(t+\sqrt{t-1})^{-1}$; then:

$$\begin{aligned} \frac{dy}{dt} &= (-1)(-1)(t+\sqrt{t-1})^{-2}[1+\frac{1}{2}(t-1)^{-\frac{1}{2}}(1)] \\ &= \frac{1}{(t+\sqrt{t-1})^2}\cdot\left(\frac{2\sqrt{t-1}}{2\sqrt{t-1}} + \frac{1}{2\sqrt{t-1}}\right) \\ &= \frac{2\sqrt{t-1}+1}{2\sqrt{t-1}(t+\sqrt{t-1})^2} \end{aligned}$$

14. $y = \dfrac{2}{(e^{3x}-1)^4} = 2(e^{3x}-1)^{-4}$; then: $y' = (2)(-4)(e^{3x}-1)^{-5}(3e^{3x}) = \dfrac{-24e^{3x}}{(e^{3x}-1)^5}$