

## SECTION 4.5 The Chain Rule (Differentiating Composite Functions)

### IN-SECTION EXERCISES:

#### EXERCISE 1.

1.

$$\begin{aligned}(f \circ g)(x) &:= f(g(x)) \\ &= f(-2x) = (-2x)^2 - 1 \\ &= 4x^2 - 1 \\ (g \circ f)(x) &:= g(f(x)) \\ &= g(x^2 - 1) = -2(x^2 - 1) \\ &= -2x^2 + 2\end{aligned}$$

2. One correct solution is to define  $g(x) = 2x$  and  $f(x) = (x + 1)^2$ . Then,

$$h(x) := f(g(x)) = f(2x) = (2x + 1)^2 .$$

Another correct solution is to define  $g(x) = 2x + 1$ , and  $f(x) = x^2$ . Then,

$$h(x) = f(g(x)) = f(2x + 1) = (2x + 1)^2 .$$

#### EXERCISE 2.

1. Since  $(f \circ g)(x) = x^2 + 2x + 1$ , differentiation yields  $(f \circ g)'(x) = 2x + 2$ . Thus,  $(f \circ g)'(2) = 2(2) + 2 = 6$ .
2. Since  $g(x) = x + 1$ ,  $g'(x) = 1$ . Thus,  $g'(2) = 1$ .
3. Since  $f(x) = x^2$ ,  $f'(x) = 2x$ . Also,  $g(2) = 2 + 1 = 3$ . Thus,  $f'(g(2)) = f'(3) = 2 \cdot 3 = 6$ .
4. Multiplying,  $f'(g(2)) \cdot g'(2) = 6 \cdot 1 = 6$ . Same answer!

#### EXERCISE 3.

1. Now,  $(g \circ f)(x) := g(f(x)) = g(x^2) = x^2 + 1$ . Differentiation yields  $(g \circ f)'(x) = 2x$ . Thus,  $(g \circ f)'(2) = 2(2) = 4$ .
2. We must have
$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) ;$$
in particular,
$$(g \circ f)'(2) = g'(f(2)) \cdot f'(2) .$$
Since  $f(x) = x^2$ ,  $f'(x) = 2x$ . Thus,  $f'(2) = 2 \cdot 2 = 4$ .
3. Since  $g(x) = x + 1$ ,  $g'(x) = 1$ . Also,  $f(2) = 2^2 = 4$ . Thus,  $g'(f(2)) = g'(4) = 1$ . (Observe that we didn't really need to find  $f(2)$ , since  $g'(\text{anything}) = 1$ .)
4. Multiplying,  $g'(f(2)) \cdot f'(2) = 1 \cdot 4 = 4$ . Same answer!

#### EXERCISE 4.

Method I: First find the function  $f \circ g$ :

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = f(3x^2 - 2x) = (3x^2 - 2x)^3 \\ &= (3x^2)^3 + 3(3x^2)^2(-2x) + 3(3x^2)(-2x)^2 + (-2x)^3 \\ &= 27x^6 - 54x^5 + 36x^4 - 8x^3\end{aligned}$$

Pascal's Triangle was used to help expand  $(3x^2 - 2x)^3$ .

Then, differentiation yields

$$(f \circ g)'(x) = 162x^5 - 270x^4 + 144x^3 - 24x^2 .$$

Method II: By the chain rule,  $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ .

$$\begin{aligned} f'(x) &= 3x^2 \\ f'(g(x)) &= f'(3x^2 - 2x) = 3(3x^2 - 2x)^2 \\ g'(x) &= 6x - 2 \\ f'(g(x)) \cdot g'(x) &= 3(3x^2 - 2x)^2(6x - 2) \\ &= 3(9x^4 - 12x^3 + 4x^2)(6x - 2) \\ &= (27x^4 - 36x^3 + 12x^2)(6x - 2) \\ &= 162x^5 - 54x^4 - 216x^4 + 72x^3 + 72x^3 - 24x^2 \\ &= 162x^5 - 270x^4 + 144x^3 - 24x^2 \end{aligned}$$

The answers agree! Normally, there is no advantage to multiplying out the expression  $3(3x^2 - 2x)^2(6x - 2)$  that one gets from using the Chain Rule. It was only multiplied out here to compare with the first answer.

#### EXERCISE 5.

$$(a \circ b \circ c \circ d \circ e)'(x) = a'(b(c(d(e(x)))))) \cdot b'(c(d(e(x)))) \cdot c'(d(e(x))) \cdot d'(e(x)) \cdot e'(x)$$

Here,  $e$  must be differentiable at  $x$ ;

$d$  must be differentiable at  $e(x)$ ;

$c$  must be differentiable at  $d(e(x))$ ;

$b$  must be differentiable at  $c(d(e(x)))$ ;

$a$  must be differentiable at  $b(c(d(e(x))))$ .

#### EXERCISE 6.

- Method I: Write  $y$  as a function of  $x$ , and differentiate.

$$\begin{aligned} y &= 3u = 3(x^2 - 1) = 3x^2 - 3 \\ \frac{dy}{dx} &= 6x \end{aligned}$$

Method II: Use the Chain Rule.

$$\frac{dy}{du} = 3, \quad \frac{du}{dx} = 2x$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 3(2x) = 6x \end{aligned}$$

- $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

3. Method I: Write  $y$  as a function of  $x$ , and differentiate.

$$y = u^2 = (3v)^2 = 9v^2 = 9(x^3)^2 = 9x^6$$

$$\frac{dy}{dx} = 54x^5$$

Method II: Use the Chain Rule.

$$\frac{dy}{du} = 2u, \quad \frac{du}{dv} = 3, \quad \frac{dv}{dx} = 3x^2$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} \\ &= 2u \cdot 3 \cdot 3x^2 = 18 \cdot u \cdot x^2 \\ &= 18 \cdot (3v) \cdot x^2 = 54 \cdot v \cdot x^2 \\ &= 54 \cdot x^3 \cdot x^2 = 54x^5 \end{aligned}$$

The answers agree!

#### EXERCISE 7.

Here are some of the things that must be ‘worried about’:

- $g$  must be differentiable at  $x$  (so that  $g'(x)$  makes sense);
- both  $(g(x))^n$  and  $n(g(x))^{n-1}$  must be defined; in particular, one must watch out for division by zero (in the case where  $n - 1$  is negative); and
- watch out for even roots of negative numbers.

#### EXERCISE 8.

1.  $f'(x) = 7(2x + 1)^6 \cdot 2 = 14(2x + 1)^6$ ;  
 $f'(0) = 14(1)^6 = 14$ ;  $f'(1) = 14(2 + 1)^6 = 14 \cdot 3^6$ .
2. First, rewrite  $f$ :  $f(x) = -(x^2 + 3)^{-1/2}$ .

Then,

$$\begin{aligned} f'(x) &= (-1)\left(-\frac{1}{2}\right)(x^2 + 3)^{-\frac{1}{2}-1} \cdot (2x) = \frac{1}{2}(x^2 + 3)^{-\frac{3}{2}} \cdot 2x \\ &= \frac{x}{(x^2 + 3)^{\frac{3}{2}}} = \frac{x}{\sqrt{(x^2 + 3)^3}}. \end{aligned}$$

Note that  $f'$  was rewritten in a form that closely resembles the original form of  $f$ .

Now,  $f'(0) = 0$  (the numerator is 0); and

$$f'(1) = \frac{1}{\sqrt{(1^2 + 3)^3}} = \frac{1}{\sqrt{4^3}} = \frac{1}{(\sqrt{4})^3} = \frac{1}{8}.$$

3. Note how the  $\frac{d}{dx}$  operator is used below:

$$\begin{aligned} f'(x) &= 3(g(h(x)))^2 \frac{d}{dx}(g(h(x))) && \text{(general power rule)} \\ &= 3(g(h(x)))^2 \cdot g'(h(x)) \cdot h'(x) && \text{(chain rule)} \end{aligned}$$

Then,  $f'(0) = 3(g(h(0)))^2 \cdot g'(h(0)) \cdot h'(0)$ . This expression cannot be simplified further without additional information about the functions  $g$  and  $h$ . In particular, remember that  $h(0)$  is the function  $h$ , *evaluated at 0*, (NOT  $h$  times 0).

Also,  $f'(1) = 3(g(h(1)))^2 \cdot g'(h(1)) \cdot h'(1)$ .

4. Note how convenient the  $\frac{d}{dx}$  operator is for intermediate steps:

$$\begin{aligned} f'(x) &= (-3)[x + (x^2 - 1)^{-2}]^{-3-1} \cdot \frac{d}{dx}(x + (x^2 - 1)^{-2}) && \text{(general power rule)} \\ &= -3[x + (x^2 - 1)^{-2}]^{-4} \cdot (1 + (-2)(x^2 - 1)^{-2-1} \cdot (2x)) && \text{(derivative of sum, general power rule)} \\ &= -3[x + (x^2 - 1)^{-2}]^{-4} \cdot (1 - 4x(x^2 - 1)^{-3}) \end{aligned}$$

Then,  $f'(0) = -3[(-1)^{-2}]^{-4} = -3$ . The functions  $f$  and  $f'$  are not defined at  $x = 1$ , since substitution into  $(x^2 - 1)^{-2}$  would cause division by zero.

#### EXERCISE 9.

Define  $f(x) = \ln x$ , so that  $(f \circ g)(x) = f(g(x)) = \ln(g(x))$ .

Recall that  $f'(x) = \frac{1}{x}$ . Then,

$$\begin{aligned} (f \circ g)'(x) &= f'(g(x)) \cdot g'(x) \\ &= \frac{1}{g(x)} \cdot g'(x) . \end{aligned}$$

In order for  $\ln(g(x))$  to be defined,  $g(x)$  must be positive. Also,  $g$  must be differentiable at  $x$ , so that  $g'(x)$  is defined. (Note: it is also true that  $\frac{d}{dx}(\ln |g(x)|) = \frac{1}{g(x)} \cdot g'(x)$ , wherever  $g$  is differentiable and  $g(x) \neq 0$ . Try to prove it!)

#### END-OF-SECTION EXERCISES:

1.  $f(x) = 2(e^x - 1)^{-1/2} + x$ ; then,

$$\begin{aligned} f'(x) &= 2\left(-\frac{1}{2}\right)(e^x - 1)^{-\frac{1}{2}-1}(e^x) + 1 \\ &= -e^x(e^x - 1)^{-\frac{3}{2}} + 1 = \frac{-e^x}{\sqrt{(e^x - 1)^3}} + 1 . \end{aligned}$$

2.  $g(x) = \sqrt[3]{x^2 - 1} = (x^2 - 1)^{1/3}$ ; then,

$$\begin{aligned} g'(x) &= \frac{1}{3}(x^2 - 1)^{\frac{1}{3}-1}(2x) \\ &= \frac{2}{3}x(x^2 - 1)^{-\frac{2}{3}} = \frac{2x}{3\sqrt[3]{(x^2 - 1)^2}} . \end{aligned}$$

3.  $y = (e^x)^3 = e^{3x}$ ; then,  $\frac{dy}{dx} = e^{3x} \cdot 3 = 3e^{3x}$ .  
 4.  $y' = 3e^{3x}$   
 5.  $y' = 11(3t - 4)^{10} \cdot 3 = 33(3t - 4)^{10}$

6.  $\frac{dy}{dt} = 8(2-t)^7(-1) = -8(2-t)^7$

7.  $g(t) = 3\sqrt[6]{t^2+t+1} = 3(t^2+t+1)^{1/6}$ ; then,

$$\begin{aligned} g'(t) &= 3\left(\frac{1}{6}\right)(t^2+t+1)^{\frac{1}{6}-1} \cdot (2t+1) \\ &= \frac{1}{2}(t^2+t+1)^{-\frac{5}{6}}(2t+1) \\ &= \frac{2t+1}{2\sqrt[6]{t^2+t+1}^5} . \end{aligned}$$

8.  $h(t) = -\sqrt[3]{\frac{1}{t^2-1}} = -\frac{1}{\sqrt[3]{t^2-1}} = -(t^2-1)^{-\frac{1}{3}}$ ; then,

$$h'(t) = (-1)\left(-\frac{1}{3}\right)(t^2-1)^{-\frac{1}{3}-1}(2t) = \frac{2t}{3}(t^2-1)^{-\frac{4}{3}} = \frac{2t}{3\sqrt[3]{(t^2-1)^4}} .$$

9.  $f'(y) = 7e^{-y}(-1) + \frac{1}{-y}(-1) = -7e^{-y} + \frac{1}{y}$

10.  $g(y) = \ln \sqrt[3]{-y} = \ln(-y)^{\frac{1}{3}} = \frac{1}{3} \ln(-y)$ ; then,  $g'(y) = \frac{1}{3} \left(\frac{1}{-y}\right)(-1) = \frac{1}{3y}$ .

11.  $\frac{dy}{dx} = 3(\ln x)^2 \cdot \frac{1}{x} = \frac{3}{x}(\ln x)^2$

12.  $y = \ln(\sqrt{x}(x+1)) = \ln \sqrt{x} + \ln(x+1) = \frac{1}{2} \ln x + \ln(x+1)$ ; then  $\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{x} + \frac{1}{x+1}(1) = \frac{1}{2x} + \frac{1}{x+1}$

13.  $y = \frac{-1}{t+\sqrt{t-1}} = -(t+\sqrt{t-1})^{-1}$ ; then

$$\begin{aligned} \frac{dy}{dt} &= (-1)(-1)(t+\sqrt{t-1})^{-2} \left[1 + \frac{1}{2}(t-1)^{-\frac{1}{2}}(1)\right] \\ &= \frac{1}{(t+\sqrt{t-1})^2} \cdot \left(\frac{2\sqrt{t-1}}{2\sqrt{t-1}} + \frac{1}{2\sqrt{t-1}}\right) \\ &= \frac{2\sqrt{t-1}+1}{2\sqrt{t-1}(t+\sqrt{t-1})^2} . \end{aligned}$$

14.  $y = \frac{2}{(e^{3x}-1)^4} = 2(e^{3x}-1)^{-4}$ ; then  $y' = (2)(-4)(e^{3x}-1)^{-5}(3e^{3x}) = \frac{-24e^{3x}}{(e^{3x}-1)^5}$