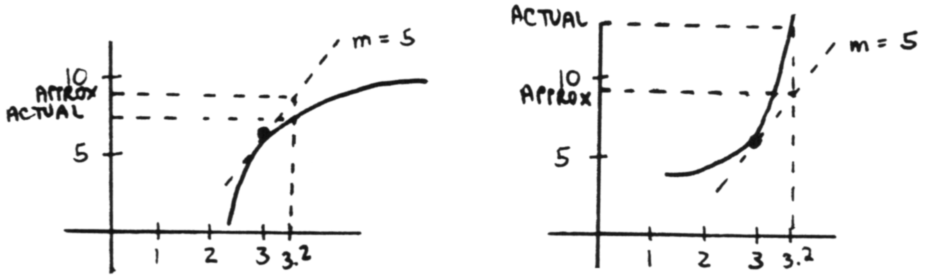
SECTION 4.4 Instantaneous Rates of Change

IN-SECTION EXERCISES:

EXERCISE 1.

1. $f(3.2) \approx 7 + (.2)(5) = 8$;
   $f(2.9) \approx 7 + (-.1)(5) = 6.5$.
2. Two possible curves are sketched below.



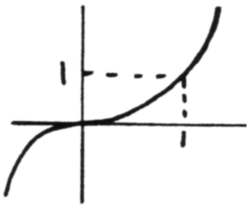
3. When $f(x) = x^2 - x + 1$, we have $f(3) = 3^2 - 3 + 1 = 7$, so the point $(3, 7)$ lies on the graph of f. Also, $f'(x) = 2x - 1$, so that $f'(3) = 2(3) - 1 = 5$. Thus, the slope of the tangent line at this point is 5.
4. The actual values of $f(3.2)$ and $f(2.9)$ are:

$$f(3.2) = (3.2)^2 - 3.2 + 1 = 8.04$$
$$f(2.9) = (2.9)^2 - 2.9 + 1 = 6.51 \ .$$

   The error at $x = 3.2$ is $|8.04 - 8| = .04$.
   The error at $x = 2.9$ is $|6.51 - 6.5| = .01$.

EXERCISE 2.

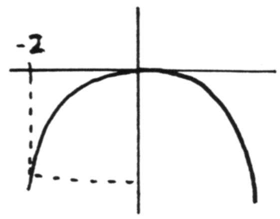
1. From $x = 1$ to $x = 2$: $\frac{f(2)-f(1)}{2-1} = \frac{8-1}{1} = 7$.
2. from $x = 1$ to $x = 1.5$: $\frac{f(1.5)-f(1)}{1.5-1} = \frac{3.375-1}{.5} = 4.75$.
3. from $x = 1$ to $x = 1.2$: $\frac{f(1.2)-f(1)}{1.2-1} = \frac{1.728-1}{.2} = 3.64$.
4. $f'(x) = 3x^2$; the instantaneous rate of change at $x = 1$ is $f'(1) = 3(1)^2 = 3$.
5. A quick sketch of the graph of the f shows why all the average rates of change were *higher*; the slopes of the tangent lines are all *greater than* 3 to the right of $x = 1$. So, actually, the rate of change of f is *faster than* 3 on any interval of the form $[1, 1 + \Delta x]$.



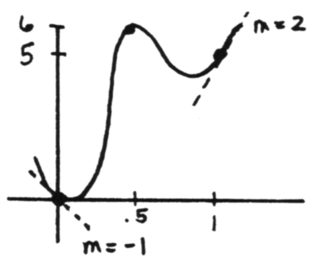
EXERCISE 3.

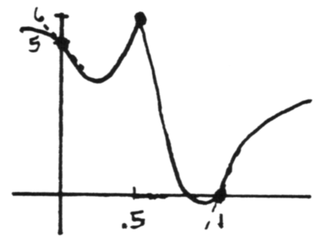
1. From $x = -2$ to $x = -1$: $\frac{f(-1)-f(-2)}{-1-(-2)} = \frac{-1-(-4)}{1} = 3$.
2. From $x = -2$ to $x = -1.5$: $\frac{f(-1.5)-f(-2)}{-1.5-(-2)} = \frac{-2.25-(-4)}{.5} = 3.5$.
3. From $x = -2$ to $x = -1.8$: $\frac{f(-1.8)-f(-2)}{-1.8-(-2)} = \frac{-3.24-(-4)}{.2} = 3.8$.
4. $f'(x) = -2x$; when $x = -2$, the instantaneous rate of change is $f'(-2) = (-2)(-2) = 4$.

5. A quick sketch of the graph of the f shows why all the average rates of change were *lower*; the slopes of the tangent lines are all *less than* 4 to the right of -2. Thus, the rate of change of f is *slower than* 4 on any interval of the form $[-2, -2 + \Delta x]$.



EXERCISE 4.
Two different graphs satisfying the desired properties are shown below.





EXERCISE 5.

1. The dummy variable is x; it represents an input that is getting closer and closer to c.
2. $\lim_{x\to c} f(x) = f(c) \quad \iff \quad \lim_{y\to c} f(y) = f(c)$.
3. f is continuous at x if and only if $\lim_{y\to x} f(y) = f(x)$.
4. Suppose the sentence $\lim\limits_{h\to 0}\big(f(x+h) - f(x)\big) = 0$ is true. Then, whenever h is close to 0, so is $f(x+h) - f(x)$. But if $f(x+h) - f(x)$ is close to 0, then $f(x+h)$ is close to $f(x)$.

   Also, whenever h is close to 0, $x + h$ is close to x.

   Combining results, whenever $x + h$ is close to x, then $f(x + h)$ is close to $f(x)$. That is, whenever the inputs to f are close to x, the corresponding outputs are close to $f(x)$. Thus, f is continuous at x.

EXERCISE 6.

1. The hypothesis is that f is differentiable at x.
2. The hypothesis was used in going from

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot h$$

   to

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h\to 0} h \ .$$

   The limit of a product is the product of the limits, *provided that each 'component' limit exists.* Since $\lim\limits_{h\to 0} h$ exists, and since $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ exists (by hypothesis), we were able to break the limit of the product into the product of the limits.

END-OF-SECTION EXERCISES:
In all cases, the 'predicted value' for $f(x_2)$ from known information at x_1 is given by

$$f(x_2) \approx f(x_1) + (\Delta x)(f'(x_1)) \ ,$$

where $\Delta x = x_2 - x_1$. All the sketches are given on the next page.

1. Here, $\Delta x = 2 - 1 = 1$; $\quad f(2) \approx 3 + (1)(2) = 5$.

2. Here, $\Delta x = 3 - 2 = 1$; $f(3) \approx 5 + (1)(-1) = 4$.
3. Here, $\Delta x = 4 - 3 = 1$; $f(4) \approx -1 + (1)(5) = 4$.
4. Here, $\Delta x = -4 - (-3) = -1$; $f(-4) \approx 2 + (-1)(1) = 1$.