SECTION 4.3 Some Very Basic Differentiation Formulas

IN-SECTION EXERCISES:

EXERCISE 1.

1. $f'(x) = 2x$
2. $f'(3) = 6$
3. $f'(3) = 6$
4. Here, the dummy variable t is being used, instead of x. The corresponding prime notation, using dummy variable t, is $f'(t) = 2t$.
5. $f'(3) = 6$
6. $f'(3) = 6$
7. Both $\frac{df}{dx} = 2x$ and $\frac{df}{dx}(x) = 2x$ are correct; the second is more strictly correct; the first is in more common usage.
8. Both $\frac{df}{dx}(3) = 6$ and $\frac{df}{dx}|_{x=3} = 6$ are correct.
9. Both $\frac{df}{dt} = 2t$ and $\frac{df}{dt}(t) = 2t$ are correct.

EXERCISE 2.

If $f(x) = \sqrt{\pi^2 - 5}$, then $\frac{df}{dx} = 0$.

If $y = e - 3$, then $y' = 0$.

To rewrite the next example, it must first be given a name:

If $y = \frac{\sqrt{7}}{3\sqrt{2}}$, then $y' = 0$.

If $f(x) = a + b$, where a and b are constants, then $\frac{df}{dx} = 0$.

EXERCISE 3.

1. Be sure to take a blank piece of paper, and prove the result *without looking at your text.* If you get stuck, study the text, but then *close your book again and prove the result yourself.* This process may need to be repeated several times before you are able to write down the entire proof yourself, correctly.
2. The limit of a sum is equal to the sum of the limits, provided that each 'component' limit exists. In the previous proof, the hypotheses state that both f and g are differentiable at x. This tells us that the limits
$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$
both exist. Since these component limits exist, we were able to write
$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \ .$$
3. Let f and g be differentiable at x. Then,
$$\begin{aligned} \lim_{h\to 0} \frac{(f-g)(x+h) - (f-g)(x)}{h} &= \lim_{h\to 0} \frac{f(x+h) - g(x+h) - \big(f(x) - g(x)\big)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) - g'(x) \ . \end{aligned}$$

EXERCISE 4.

$$\begin{aligned}(f+g+h+k)'(x) &= \big((f+g)+(h+k)\big)'(x) && \text{(group)}\\ &= (f+g)'(x)+(h+k)'(x) && \text{(use result once)}\\ &= f'(x)+g'(x)+h'(x)+k'(x) && \text{(use result again)}\end{aligned}$$

Other groupings could also be used!

EXERCISE 5.

1. First, rewrite: $f(x)=\sqrt[3]{x}=x^{1/3}$. Using the simple power rule,

$$\begin{aligned}f'(x) &= \frac{1}{3}x^{\frac{1}{3}-1}=\frac{1}{3}x^{-\frac{2}{3}}\\ &= \frac{1}{3}\cdot\frac{1}{x^{2/3}}=\frac{1}{3}\cdot\frac{1}{(x^2)^{\frac{1}{3}}}\\ &= \frac{1}{3}\cdot\frac{1}{\sqrt[3]{x^2}}=\frac{1}{3\sqrt[3]{x^2}}\ .\end{aligned}$$

The expression $\sqrt[3]{x}$ is defined for all real numbers x; the expression $\frac{1}{3\sqrt[3]{x^2}}$ is defined for all nonzero real numbers. BOTH expressions are defined on $\mathbb{R}-\{0\}$, so the derivative formula is valid for all nonzero real numbers. (There is a *vertical* tangent line at $x=0$.)

When $x=1$, $f(1)=\sqrt[3]{1}=1$, so the point $(1,1)$ lies on the graph of f. The slope of the tangent line here is given by $f'(1)=\frac{1}{3\sqrt[3]{1^2}}=\frac{1}{3}$. The equation of the tangent line to the graph of f when $x=1$ is

$$y-1=\frac{1}{3}(x-1)\ .$$

2. First, rewrite: $f(x)=\frac{1}{\sqrt{x}}=\frac{1}{x^{1/2}}=x^{-1/2}$. Using the simple power rule,

$$\begin{aligned}f'(x) &= -\frac{1}{2}x^{-\frac{1}{2}-1}=-\frac{1}{2}x^{-\frac{3}{2}}\\ &= -\frac{1}{2}\cdot\frac{1}{x^{3/2}}=-\frac{1}{2}\cdot\frac{1}{(x^3)^{\frac{1}{2}}}\\ &= -\frac{1}{2}\cdot\frac{1}{\sqrt{x^3}}=-\frac{1}{2\sqrt{x^3}}\ .\end{aligned}$$

The expression $\frac{1}{\sqrt{x}}$ is defined for all positive real numbers x, as is the expression $-\frac{1}{2\sqrt{x^3}}$. Thus, the derivative formula is valid for all positive real numbers.

When $x=1$, $f(1)=\frac{1}{\sqrt{1}}=1$, so the point $(1,1)$ lies on the graph of f. The slope of the tangent line here is given by $f'(1)=-\frac{1}{2\sqrt{1^3}}=-\frac{1}{2}$. The equation of the tangent line to the graph of f when $x=1$ is

$$y-1=-\frac{1}{2}(x-1)\ .$$

3. First, rewrite:

$$f(x) = \sqrt{x}\sqrt[3]{x^2} = x^{1/2} \cdot (x^2)^{1/3} = x^{1/2}x^{2/3} = x^{\frac{1}{2}+\frac{2}{3}} = x^{\frac{3}{6}+\frac{4}{6}} = x^{\frac{7}{6}} \ .$$

Using the simple power rule,

$$f'(x) = \frac{7}{6}x^{\frac{7}{6}-1} = \frac{7}{6}x^{\frac{1}{6}} = \frac{7}{6}\sqrt[6]{x} \ .$$

The expression $\sqrt{x}\sqrt[3]{x^2}$ is defined for all nonnegative real numbers x, as is the expression $\frac{7}{6}\sqrt[6]{x}$. Thus, the derivative formula is valid on the interval $[0, \infty)$.

When $x = 1$, $f(1) = 1$, so (again!) the point $(1, 1)$ lies on the graph of f. The slope of the tangent line here is $f'(1) = \frac{7}{6}\sqrt[6]{1} = \frac{7}{6}$. The equation of the tangent line to the graph of f when $x = 1$ is

$$y - 1 = \frac{7}{6}(x - 1) \ .$$

EXERCISE 6.

1. The term 'types' are:

$$x^9 \quad x^8h \quad x^7h^2 \quad x^6h^3 \quad x^5h^4 \quad x^4h^5 \quad x^3h^6 \quad x^2h^7 \quad xh^8 \quad h^9 \ .$$

The coefficients come from the row of Pascal's triangle that begins with '1 9 ':

$$(x + h)^9 = x^9 + 9x^8h + 36x^7h^2 + 84x^6h^3 + 126x^5h^4 + 126x^4h^5 + 84x^3h^6 + 36x^2h^7 + 9xh^8 + h^9 \ .$$

2.

$$(x - h)^4 = x^4 + 4x^3(-h) + 6x^2(-h)^2 + 4x(-h)^3 + (-h)^4 = x^4 - 4x^3h + 6x^2h^2 - 4xh^3 + h^4 \ .$$

3.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} \\ &= \lim_{h\to 0} \frac{(x^4 + \overbrace{4x^3h}^{\text{one factor of } h} + \overbrace{6x^2h^2 + 4xh^3}^{\text{more than one } h}) - x^4}{h} \\ &= \lim_{h\to 0} \frac{h(4x^3 + 6x^2h + 4xh^2)}{h} \\ &= \lim_{h\to 0} (4x^3 + 6x^2h + 4xh^2) \\ &= 4x^3 \ . \end{aligned}$$

EXERCISE 7.

1. First, rewrite: $f(x) = e^{x+5} = e^xe^5 = (e^5) \cdot e^x$. Then,

$$f'(x) = e^5 \cdot \frac{d}{dx}(e^x) = e^5 \cdot e^x = e^{5+x} = e^{x+5} \ .$$

Thus, $f'(x) = f(x)$. Again, the y-value of the point on the graph of f tells us the slope of the tangent line at that point!

2. First, rewrite: $f(x) = \ln 7x = \ln 7 + \ln x$. Then,

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\ln 7) + \frac{d}{dx}(\ln x) \\ &= 0 + \frac{1}{x} = \frac{1}{x} . \end{aligned}$$

3. Although the function $f(x) = e^{2x}$ can be rewritten as $f(x) = (e^x)^2$, this doesn't help us to differentiate it. We need to know how to differentiate a FUNCTION to a power. As soon as learn how to differentiate composite functions, we'll be able to (easily) differentiate e^{2x}.
4. There is no easy way to rewrite the log of a sum. Again, this problem must be postponed until we know how to differentiate composite functions.

EXERCISE 8.

1. The lead-in phrase, 'For all real numbers x and y,' informs the reader of the universal sets for the variables x and y. That is, in the remainder of the sentence, x and y are allowed to be *any* real numbers.

   The two sentences being compared in (*) are '$y = \sqrt[3]{x}$' and '$y^3 = x$'. These sentences are equivalent; thus, no matter what real numbers are substituted in for x and y, the sentences will have the SAME truth values.

   Indeed, the sentence '$y = \sqrt[3]{x}$' is being *defined*; the reader is being told that whenever the sentence '$y^3 = x$' is true, so is '$y = \sqrt[3]{x}$'; and whenever the sentence '$y^3 = x$' is false, so is '$y = \sqrt[3]{x}$'.

   When $y = 2$ and $x = 8$, the sentence '$y^3 = x$' becomes '$2^3 = 8$', which is true, hence so is the sentence '$2 = \sqrt[3]{8}$'.

   When $y = -2$ and $x = 8$, the sentence '$y^3 = x$' becomes '$(-2)^3 = 8$', which is false, hence so is the sentence '$-2 = \sqrt[3]{8}$'.

2. The lead-in phrase, 'For all $x \geq 0$ and for all real numbers y,' informs the reader of the universal sets for the variables x and y. That is, in the remainder of the sentence, x represents a nonnegative number, and y is *any* real number.

   The two sentences being compared in (**) are '$y = \sqrt{x}$' and '$y \geq 0$ and $y^2 = x$'. These sentences are equivalent; thus, no matter what numbers are substituted in for x and y from their universal sets, the sentences will have the SAME truth values.

   Indeed, the sentence '$y = \sqrt{x}$' is being *defined*; the reader is being told that whenever the sentence '$y \geq 0$ and $y^2 = x$' is true, so is '$y = \sqrt{x}$'; and whenever the sentence '$y \geq 0$ and $y^2 = x$' is false, so is '$y = \sqrt{x}$'.

   When $y = 2$ and $x = 4$, the sentence '$y \geq 0$ and $y^2 = x$' becomes '$2 \geq 0$ and $2^2 = 4$', which is true, hence so is the sentence '$2 = \sqrt{4}$'.

   When $y = -2$ and $x = 8$, the sentence '$y \geq 0$ and $y^2 = x$' becomes '$-2 \geq 0$ and $(-2)^2 = 4$', which is false, hence so is the sentence '$-2 = \sqrt{4}$'. (If necessary, review the mathematical meaning of the word 'and', from Chapter 1.)

3. $\sqrt[5]{-32} = -2$, since $(-2)^5 = -32$.
4. $\sqrt[4]{(-2)^4} = 2$, since $2 \geq 0$ and $2^4 = (-2)^4$.
5. $\sqrt[6]{x^6} = |x|$, since $|x| \geq 0$ and $(|x|)^6 = x^6$.
6. $\sqrt[9]{x^9} = x$, since $(x)^9 = x^9$.

EXERCISE 9.
To illustrate the idea behind $\frac{a^m}{a^n} = a^{m-n}$, suppose that $m > n$ and $a \neq 0$, and write

$$\begin{aligned}
\frac{a^m}{a^n} &= \frac{\overbrace{a \cdot \ldots \cdot a}^{m \text{ factors of } a}}{\underbrace{a \cdot \ldots \cdot a}_{n \text{ factors of } a}} \\
&= \overbrace{\left(\frac{a \cdot \ldots \cdot a}{a \cdot \ldots \cdot a}\right)}^{n \text{ factors of } 1} \cdot \overbrace{\left(\frac{a \cdot \ldots \cdot a}{1}\right)}^{m-n \text{ factors of } a} \\
&= \frac{a^{m-n}}{1} = a^{m-n} \ .
\end{aligned}$$

Have fun with the rest!

EXERCISE 10.
1. Roughly, the sentence '$\ln \frac{a}{b} = \ln a - \ln b$' says that the log of a quotient is the difference of the logs. Let $a > 0$ and $b > 0$. Then,

$$\begin{aligned}
y = \ln a - \ln b \quad &\iff \quad e^y = e^{\ln a - \ln b} && (e^x \text{ is a } 1-1 \text{ function}) \\
&\iff \quad e^y = \frac{e^{\ln a}}{e^{\ln b}} && (\text{properties of exponents, } e^x \neq 0) \\
&\iff \quad e^y = \frac{a}{b} && (e^{\ln a} = a \text{ and } e^{\ln b} = b) \\
&\iff \quad \ln e^y = \ln \frac{a}{b} && (\ln x \text{ is a } 1-1 \text{ function}) \\
&\iff \quad y = \ln \frac{a}{b} \ . && (\ln e^y = y)
\end{aligned}$$

Thus, the sentences $y = \ln a - \ln b$ and $y = \ln \frac{a}{b}$ always have the same truth values. That is, $\ln \frac{a}{b} = \ln a - \ln b$.

2. Let $a > 0$; b can be any real number. Then,

$$\begin{aligned}
y = b \ln a \quad &\iff \quad e^y = e^{b \ln a} && (e^x \text{ is a } 1-1 \text{ function}) \\
&\iff \quad e^y = (e^{\ln a})^b && (\text{properties of exponents}) \\
&\iff \quad e^y = a^b && (e^{\ln a} = a) \\
&\iff \quad \ln e^y = \ln a^b && (\ln x \text{ is a } 1-1 \text{ function}) \\
&\iff \quad y = \ln a^b \ . && (\ln e^y = y)
\end{aligned}$$

Thus, the sentences $y = \ln a^b$ and $y = b \ln a$ always have the same truth values. That is, $\ln a^b = b \ln a$.

END-OF-SECTION EXERCISES:

1. After we get the chain rule, there will be an easier way to differentiate this function. For now, we must first multiply it out, using Pascal's triangle to help:

$$\begin{aligned}(2x+1)^3 &= (1)(2x)^3 + (3)(2x)^2(1) + (3)(2x)^1(1)^2 + (1)(1)^3 \\ &= 8x^3 + 12x^2 + 6x + 1 \ .\end{aligned}$$

Thus,

$$\begin{aligned}f'(x) &= \frac{d}{dx}(8x^3 + 12x^2 + 6x + 1) \\ &= 24x^2 + 24x + 6 \\ &= 6(4x^2 + 4x + 1) \\ &= 6(2x+1)^2 \ .\end{aligned}$$

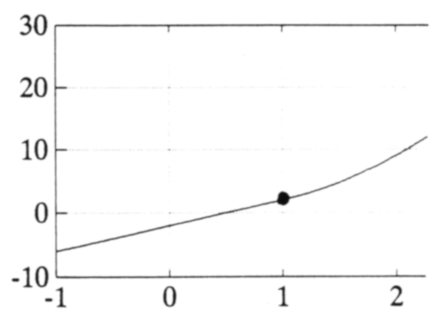
2. First, rewrite g in a more suitable form:

$$\begin{aligned}g(x) &= \frac{\sqrt{x}+1}{\sqrt[7]{x}} = \frac{\sqrt{x}}{\sqrt[7]{x}} + \frac{1}{\sqrt[7]{x}} \\ &= \frac{x^{1/2}}{x^{1/7}} + \frac{1}{x^{1/7}} = x^{\frac{1}{2}-\frac{1}{7}} + x^{-\frac{1}{7}} \\ &= x^{\frac{7}{14}-\frac{2}{14}} + x^{-\frac{1}{7}} = x^{\frac{5}{14}} + x^{-\frac{1}{7}} \ .\end{aligned}$$

Then,

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left(x^{\frac{5}{14}} + x^{-\frac{1}{7}}\right) = \frac{5}{14}x^{\frac{5}{14}-1} + (-\frac{1}{7})x^{-\frac{1}{7}-1} \\ &= \frac{5}{14}x^{-\frac{9}{14}} - \frac{1}{7}x^{-\frac{8}{7}} = \frac{5}{14\sqrt[14]{x^9}} - \frac{1}{7\sqrt[7]{x^8}} \ .\end{aligned}$$

3. A quick sketch helps. For $x \ge 1$, the graph is a (piece of) a parabola. Note that $h(1) = 3(1)^2 - 2(1) + 1 = 2$, and $\mathcal{D}(h) = \mathbb{R}$.



For $x > 1$ and $x < 1$, h is differentiable, and

$$h'(x) = \begin{cases} 6x - 2 & \text{for } x > 1 \\ 4 & \text{for } x < 1 \ . \end{cases}$$

To see if h is differentiable at 1, we could investigate two one-sided limits. Alternately, observe that:

As x approaches 1 from the right, the slopes of the tangent lines approach $6(1) - 2 = 4$.

As x approaches 1 from the left, the slopes of the tangent lines are all 4.

The 'directions' as we approach 1 from both the left and the right agree! Thus, h is also differentiable at 1, and $h'(1) = 4$. Thus, we can write

$$h'(x) = \begin{cases} 6x - 2 & \text{for } x \ge 1 \\ 4 & \text{for } x < 1 \ . \end{cases}$$

4. A quick sketch helps. The graph looks the same as in the previous question, except the slope of the tangent line for the 'left-hand piece' is 3. Still, $h(1) = 3(1)^2 - 2(1) + 1 = 2$, and $\mathcal{D}(h) = \mathbb{R}$.

   For $x > 1$ and $x < 1$, h is differentiable, and

$$h'(x) = \begin{cases} 6x - 2 & \text{for } x > 1 \\ 3 & \text{for } x < 1 \ . \end{cases}$$

   To see if h is differentiable at 1, we could investigate two one-sided limits, and show that they do NOT agree. Alternately, observe that:

   As x approaches 1 from the right, the slopes of the tangent lines approach $6(1) - 2 = 4$.

   As x approaches 1 from the left, the slopes of the tangent lines are all 3.

   The 'directions' as we approach 1 from both the left and the right do NOT agree! Thus, h is not differentiable at 1.