SECTION 4.2 The Derivative

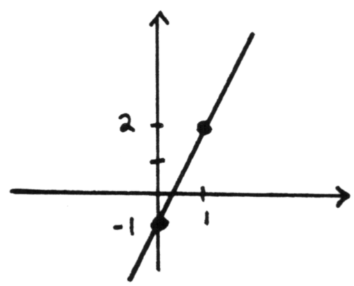
IN-SECTION EXERCISES:

EXERCISE 1.

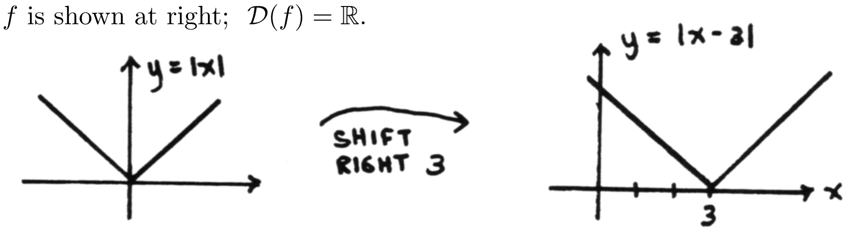
1. Set subtraction is only defined for *sets*; that is, in the expression $A - B$, both A and B must be *sets*. However, in '$\mathbb{R} - 0$', '0' is not a set, it is a number.
2. $A - B = (2, \infty)$, $B - A = \emptyset$
3. $A - B = (-3, -1)$, $B - A = (3, 4)$
4. $A - B = \mathbb{Q}$ (Remember that $\mathbb{Q}$ represents the *set* of rational numbers.)
   $B - A = \emptyset$
5. $S = (-1, 1] - \{0\}$
6. $S = \{1, 2, 3, 4\} - \{4\}$

EXERCISE 2.

1. The graph of f is shown at right; $\mathcal{D}(f) = \mathbb{R}$.



2. At *every* point, the slope of the tangent line is 3. Thus, $f'(x) = 3$. Here, $\mathcal{D}(f') = \mathbb{R}$.
3. $$f(x) = \begin{cases} x - 3 & \text{for } x \ge 3 \\ 3 - x & \text{for } x < 3 \ . \end{cases}$$
4. The graph of f is shown at right; $\mathcal{D}(f) = \mathbb{R}$.



5. For $x > 3$, $f'(x) = 1$. This is because at any point $(x, f(x))$ with $x > 3$, the slope of the tangent line is 1.
6. For $x < 3$, $f'(x) = -1$. This is because at any point $(x, f(x))$ with $x < 3$, the slope of the tangent line is -1.
7. There is no tangent line at the point $(3, 0)$. This is confirmed by investigating the one-sided limits:

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(3+h) - f(3)}{h} &= \lim_{h \to 0^+} \frac{((3+h) - 3) - 0}{h} \\ &= \lim_{h \to 0^+} \frac{h}{h} = 1 \end{aligned}$$

and

$$\begin{aligned} \lim_{h \to 0^-} \frac{f(3+h) - f(3)}{h} &= \lim_{h \to 0^+} \frac{(3 - (3+h)) - 0}{h} \\ &= \lim_{h \to 0^+} \frac{-h}{h} = -1 \ . \end{aligned}$$

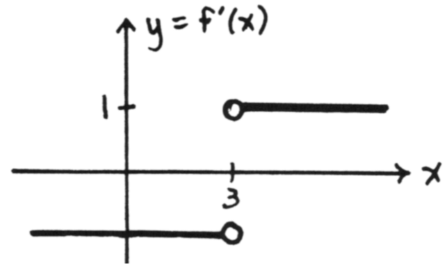
Since these one-sided limits do not agree, $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$ does not exist.

8.

$$f'(x) = \begin{cases} 1 & \text{for } \ x > 3 \\ -1 & \text{for } \ x < 3 \\ \text{not defined} & \text{for } \ x = 3 \ . \end{cases}$$

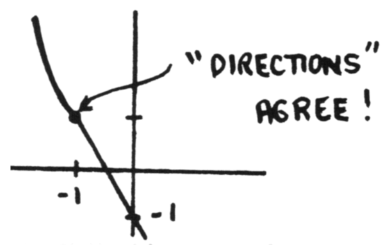
In particular, $\mathcal{D}(f') = \mathbb{R} - \{3\}$.

9. The graph of f' is shown at right.



EXERCISE 3.

1. The graph of f is shown at right; $\mathcal{D}(f) = \mathbb{R}$.



2. For $x < -1$, $f(x) = x^2$, and $f'(x) = 2x$. That is, at every point $(x, f(x))$ with $x < -1$, the tangent line exists and has slope $2x$.
3. When x is close to -1, coming in from the left-hand side, $f'(x) = 2x$. Thus,

$$\lim_{x \to -1^-} f'(x) = \lim_{x \to -1^-} 2x = 2(-1) = -2 \ .$$

That is, we can get the slope of the tangent line to the graph of f *as close to* -2 *as desired*, by requiring that x be sufficiently close to -1, coming in from the left-hand side.

4. For $x > -1$, $f(x) = -2x - 1$, and $f'(x) = -2$. That is, at every point $(x, f(x))$ with $x > -1$, the tangent line exists and has slope 2.
5. Since the function is defined differently to the left and right of -1, we must investigate two one-sided limits:

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(-1+h) - f(-1)}{h} &= \lim_{h \to 0^+} \frac{(-2(-1+h) - 1) - 1}{h} \\ &= \lim_{h \to 0^+} \frac{-2h}{h} = -2 \ . \end{aligned}$$

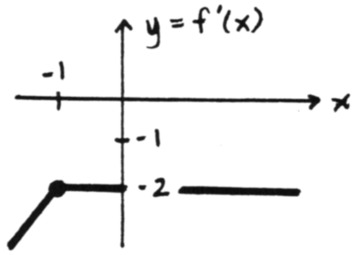
Also,

$$\begin{aligned} \lim_{h \to 0^-} \frac{f(-1+h) - f(-1)}{h} &= \lim_{h \to 0^-} \frac{(-1+h)^2 - 1}{h} \\ &= \lim_{h \to 0^-} \frac{(1 - 2h + h^2) - 1}{h} \\ &= \lim_{h \to 0^-} \frac{h(-2+h)}{h} \\ &= \lim_{h \to 0^-} (-2+h) = -2 \ . \end{aligned}$$

Since the one-sided limits agree, $\lim_{h \to 0} \dfrac{f(-1+h) - f(-1)}{h}$ exists and equals -2. That is, $f'(-1) = -2$.

6. There IS a tangent line to the graph of f at the point $(-1, 1)$; it has slope -2.

7. The graph of f' is shown below; $\mathcal{D}(f') = \mathbb{R}$.



EXERCISE 4.

1. line 1: By definition, $f(1+h) = \frac{1}{1+h}$. Also, since f is only defined for $h > 0$, the 'two-sided' limit is, in this case, just a right-hand limit.

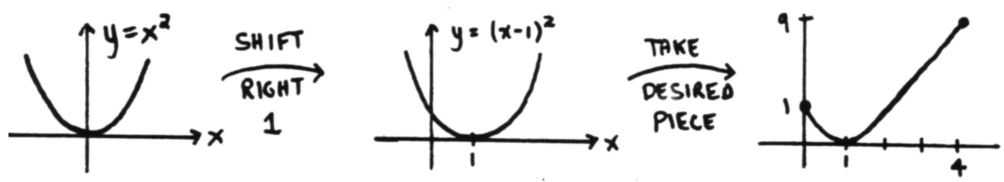
   line 2: Get a common denominator in the numerator.

   line 3: Here are some missing details:

$$\frac{\frac{1}{1+h} - \frac{1+h}{1+h}}{h} = \frac{\frac{1-(1+h)}{1+h}}{h} = \frac{1-1-h}{1+h} \cdot \frac{1}{h} = \frac{1-1-h}{h(1+h)} \ .$$

   line 4: Here, the function $\frac{-h}{h(1+h)}$ has been replaced by the function $\frac{-1}{1+h}$ that agrees with it, except when $h = 0$. This function $\frac{-1}{1+h}$ is continuous at $h = 0$, so evaluating the limit is as easy as direct substitution.

2. The graph of f is found below. Note that $f(0) = (0-1)^2 = 1$, and $f(4) = (4-1)^2 = 9$.



3. Since f is only defined to the right of 0, the 'two-sided' limit is actually a right-hand limit. Thus,

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{(h-1)^2-1}{h} = \lim_{h\to 0^+} \frac{(h^2-2h+1)-1}{h} = \lim_{h\to 0^+} \frac{h(h-2)}{h} = \lim_{h\to 0^+} (h-2) = -2 \ .$$

   Thus, f is differentiable at 0, and $f'(0) = -2$.

EXERCISE 5.

line 1: By definition of f, $f(x+h) = \sqrt{x+h}$ and $f(x) = \sqrt{x}$.

line 2: Since the limit in line (1) is an indeterminate form, it must be put in a form that is easier to analyze when h is *close to* 0 (but not equal to 0). This is accomplished, in this case, by rationalizing the numerator.

line 3: Multiply out the numerator:

$$\begin{aligned}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} &= \frac{\sqrt{x+h}\sqrt{x+h}-\sqrt{x}\sqrt{x+h}+\sqrt{x}\sqrt{x+h}-\sqrt{x}\sqrt{x}}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}\ ;\end{aligned}$$

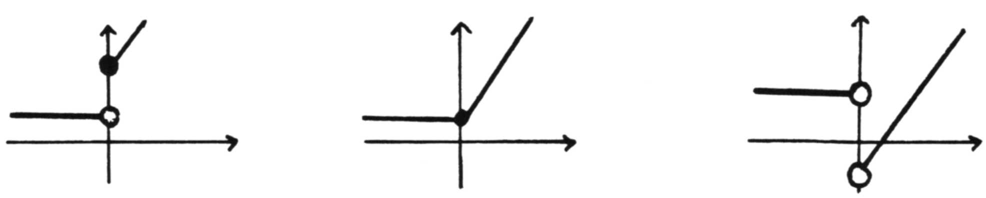
the fact that $x>0$ was used to conclude that $\sqrt{x}\sqrt{x}=x$ and $\sqrt{x+h}\sqrt{x+h}=x+h$.
line 4: Cancel h; since the functions $\frac{h}{h(\sqrt{x+h}+\sqrt{x})}$ and $\frac{1}{\sqrt{x+h}+\sqrt{x}}$ agree everywhere except at 0, and since h is not allowed to equal 0 in evaluating the limit, this is valid.
line 5: The function $\frac{1}{\sqrt{x+h}+\sqrt{x}}$ is continuous at $h=0$, so evaluating the limit is as easy as direct substitution.

EXERCISE 6.

1. $f(0)=3$; $f(1)=2$; $f'(1)$ does not exist; $f'(2)$ does not exist; $f'(1,34)=-1$ (Use the known points $(1,2)$ and $(2,1)$ to compute this slope); $f(3)$ does not exist, i.e., f is not defined at 3; $f(4)=-1$; $f'(\pi)=0$; assuming that the pattern at the right-hand border of the graph continues, estimate that $f'(1000)\approx 0$.
2. $\mathcal{D}(f)=(-\infty,3)\cup(3,\infty)=\mathbb{R}-\{3\}$
3. $\mathcal{D}(f')=(-\infty,1)\cup(1,2)\cup(2,3)\cup(3,4)\cup(4,\infty)=\mathbb{R}-\{1,2,3,4\}$
4. $\mathcal{R}(f)=\mathbb{R}$
5. The function f is continuous for all x in the set $\mathbb{R}-\{1,3,4\}$. At $x=1$, the discontinuity is nonremovable. At $x=3$, the discontinuity is removable. At $x=4$, the discontinuity is nonremovable.
6. Some approximation is necessary. We seek all points that have nonpositive y-values.
   $\{x \mid f(x)\le 0\}=[2.5,3)\cup(3,4]$
7. We seek all points with tangent lines that have a negative slope.
   $\{x \mid f'(x)<0\}=(1,2)\cup(2,3)$

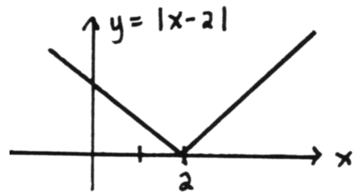
EXERCISE 7.

1. For all $x>0$, the slope of the tangent line to the graph of f must be 2. For all $x<0$, the slopes must be 0.
2. Below are shown three different functions f that meet these requirements.



END-OF-SECTION EXERCISES:

1. The graph of f is shown at right. Here, $\mathcal{D}(f) = \mathbb{R}$.

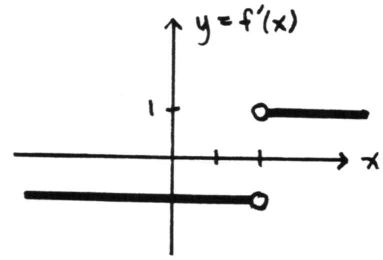


When $x > 2$, the slopes of the tangent lines equal 1.

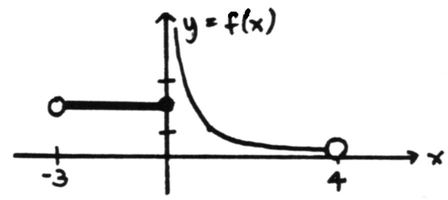
When $x < 2$, the slopes of the tangent lines equal -1.

There is no tangent line at $x = 2$.

The graph of f' is shown at right. Here, $\mathcal{D}(f') = \mathbb{R} - \{2\}$.



2. The graph of f is shown at right. Here, $\mathcal{D}(f) = (-3, 4)$.

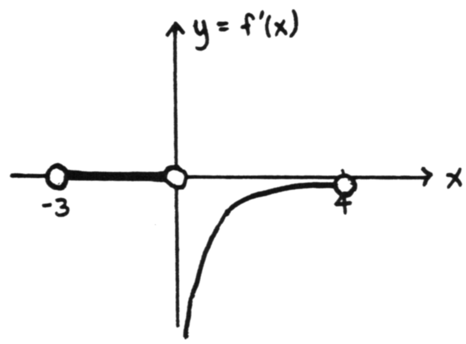


When $x \in (-3, 0)$, the slopes of the tangent lines equal 0.

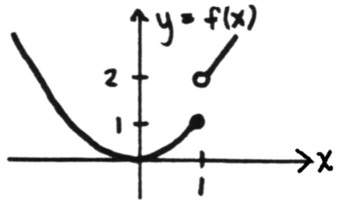
When $x \in (0, 4)$, the slopes of the tangent lines are:

$$
\begin{aligned}
f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x}{x}\frac{1}{x+h} - \frac{1}{x}\frac{x+h}{x+h}}{h} \\
&= \lim_{h\to 0} \frac{x - (x+h)}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2} \ .
\end{aligned}
$$

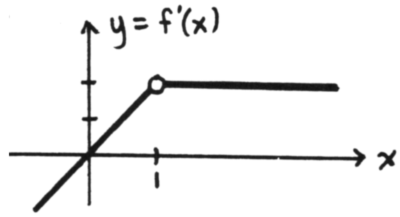
The graph of f' is shown below. Here, $\mathcal{D}(f') = (-3, 0) \cup (0, 4)$.



3. The graph of f is shown below. Here, $\mathcal{D}(f) = \mathbb{R}$.



When $x > 1$, the slopes of the tangent lines equal 2.
When $x < 1$, the slopes of the tangent lines equal $2x$ (as per an example in the text).
There is no tangent line at $x = 1$.
The graph of f' is shown below. Here, $\mathcal{D}(f') = \mathbb{R} - \{1\}$.



4. Note that $f(2) = 3(2)^2 - 1 = 11$. Then,

$$\begin{aligned}\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h\to 0} \frac{[3(2+h)^2 - 1] - 11}{h} \\ &= \lim_{h\to 0} \frac{3(4 + 4h + h^2) - 1 - 11}{h} \\ &= \lim_{h\to 0} \frac{12 + 12h + 3h^2 - 12}{h} \\ &= \lim_{h\to 0} \frac{3h(4+h)}{h} \\ &= \lim_{h\to 0} 3(4+h) = 12\ .\end{aligned}$$

Thus, $f'(2) = 12$. That is, the slope of the tangent line to the graph of f at the point $(2, 11)$ equals 12.

5. Note that $f(2) = \frac{1}{2-1} = 1$. Then,

$$\begin{aligned}\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h\to 0} \frac{\frac{1}{(2+h)-1} - 1}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{1+h} - \frac{1+h}{1+h}}{h} \\ &= \lim_{h\to 0} \frac{1 - (1+h)}{h(1+h)} \\ &= \lim_{h\to 0} \frac{-1}{1+h} = -1\ .\end{aligned}$$

Thus, $f'(2) = -1$. That is, the slope of the tangent line to the graph of f at the point $(2, 1)$ is -1.

6. Note that $f(4) = \sqrt{4} + 1 = 3$. Then,

$$\begin{aligned}\lim_{h\to 0} \frac{f(4+h) - f(4)}{h} &= \lim_{h\to 0} \frac{(\sqrt{4+h} + 1) - 3}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \\ &= \lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h} + 2)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{4} \ .\end{aligned}$$

Thus, $f'(4) = \frac{1}{4}$. That is, the slope of the tangent line to the graph of f at the point $(4, 3)$ is $\frac{1}{4}$.

7. First, evaluate

$$\lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

to show that $f'(3) = 6$. The equation of the line that passes through the point $(3, f(3)) = (3, 9)$ and has slope 6 is

$$y - 9 = 6(x - 3) \ .$$

8. The limit

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist; there is a vertical tangent line at the point $(0, 0)$. The equation of this vertical tangent line is $x = 0$.

9. First, evaluate

$$\lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h}$$

to show that $f'(-2) = 0$. There is a horizontal tangent line at the point $(-2, 1)$; its equation is $y = 1$.