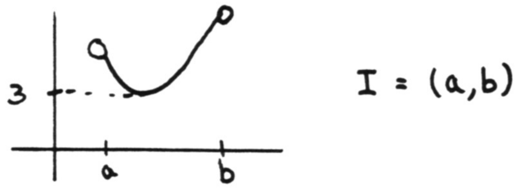
SECTION 3.7 The Max-Min Theorem

IN-SECTION EXERCISES:

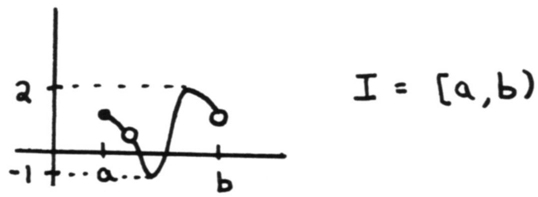
EXERCISE 1.

There are many possible correct answers.

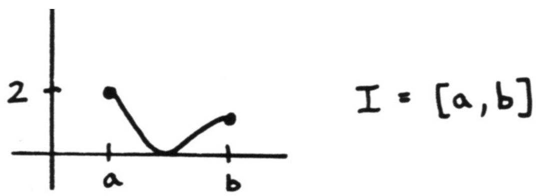
1.



2.

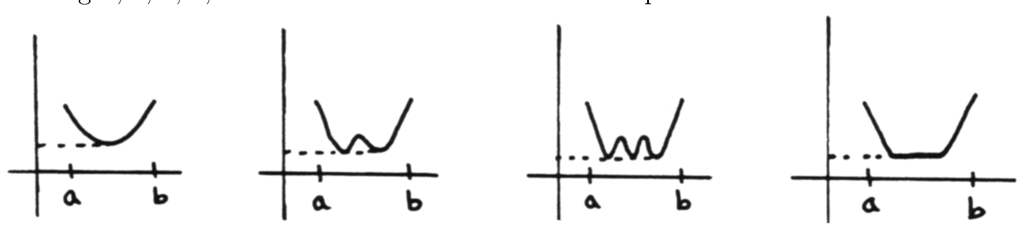


3.



EXERCISE 2.

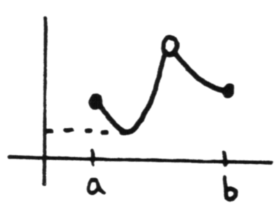
1. If a function has a minimum *value*, then it must be unique.
2. However, a minimum point certainly need NOT be unique. The functions sketched below have, going from left to right, 1, 2, 3, and an infinite number of minimum points on I.



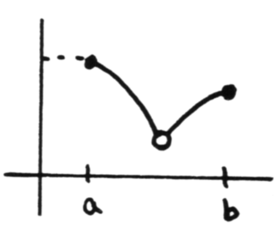
EXERCISE 3.

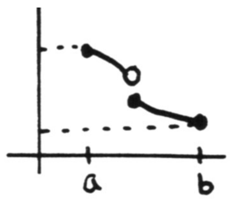
There are many possible correct answers.

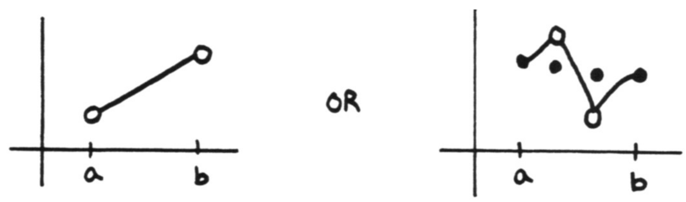
1.



2.



3. 

4. 

5. If f is NOT continuous on $[a, b]$, then the Max-Min Theorem cannot be used to reach any conclusion about extreme values of f on $[a, b]$. Indeed, f MAY or MAY NOT have extreme values, as the previous examples illustrate.

EXERCISE 4.

1. If f IS continuous on the closed interval I, then it would HAVE to attain a maximum value (by the Max-Min Theorem). Therefore, it must be that f is NOT continuous on I.
2. If f does NOT attain a maximum value on $[a, b]$, then it must NOT be continuous on $[a, b]$. However, it is known that f is defined on $[a, b]$ and continuous on (a, b). Therefore, it must be that f 'goes bad' at an endpoint. This is illustrated below.

EXERCISE 5.

1. TRUE. Whenever x is a number in the interval $[1, 2]$, then x is positive.

   Contrapositive: If $x \le 0$, then $x \notin [1, 2]$

   Alternately: If $x \le 0$, then $x \in (-\infty, 1) \cup (2, \infty)$

2. FALSE. Let $x = 0$. Then the hypothesis '$0 \in [0, 1)$' is TRUE, but the conclusion '$0 > 0$' is false.

   Contrapositive: If $x \le 0$, then $x \notin [0, 1)$

   Alternately: If $x \le 0$, then $x \in (-\infty, 0) \cup [1, \infty)$

3. TRUE. Whenever x is a number in the interval $[0, 1)$, then x is a number that is greater than or equal to 0.

   Contrapositive: If $x < 0$, then $x \in (-\infty, 0) \cup [1, \infty)$

4. TRUE. This is a consequence of the Max-Min Theorem.

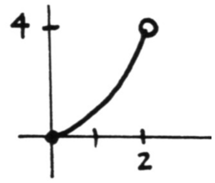
   Contrapositive: If f does not attain a minimum value on $[a, b]$, then f is not continuous on $[a, b]$.

5. TRUE. This is a consequence of the Intermediate Value Theorem.

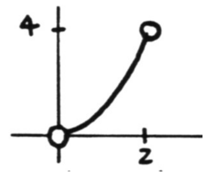
   Contrapositive: If there does NOT exist a number $c \in [a, b]$ with $f(c) = D$, then f is not continuous on $[a, b]$. (There are other correct ways to state this.)

END-OF-SECTION EXERCISES:

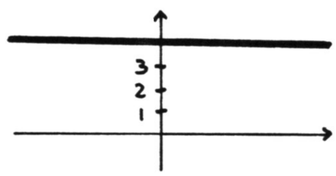
1. The minimum value of f on I is 0; there is no maximum value. The only minimum point is $(0, 0)$.



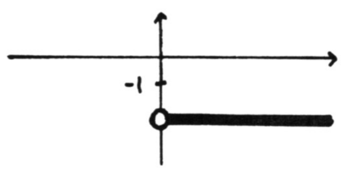
2. There is no minimum or maximum value.



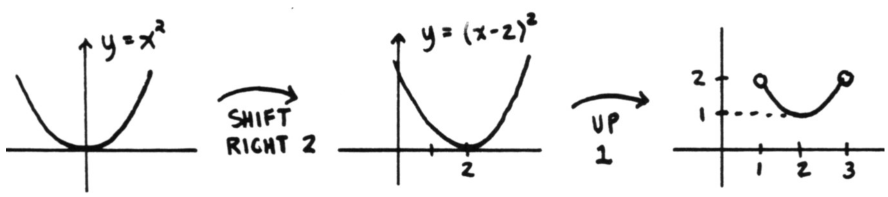
3. The maximum value of f on I is 4; the minimum value of f on I is 4. The points $(x, 4)$ for $x \in \mathbb{R}$ are all both maximum and minimum points.



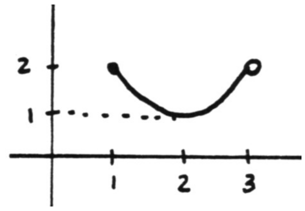
4. The maximum and minimum value is -2; the points $(x, -2)$ for $x \in I$ are all both maximum and minimum points.



5. The minimum value of f on I is 1; there is no maximum value. The point $(2, 1)$ is the only minimum point.



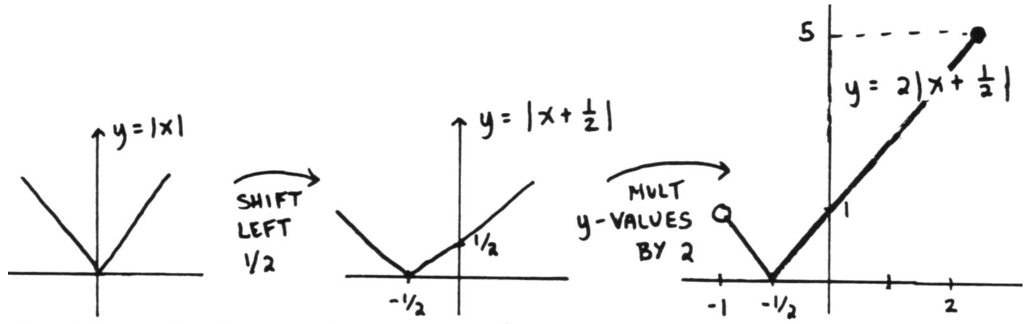
6. The minimum value of f on I is 1; the maximum value is 2. The point $(2, 1)$ is the only minimum point; the point $(1, 2)$ is the only maximum point.



7. There are two nice ways to sketch the graph of $f(x) = |2x+1|$. One way is illustrated here; the next way in the problem (8). First, write:

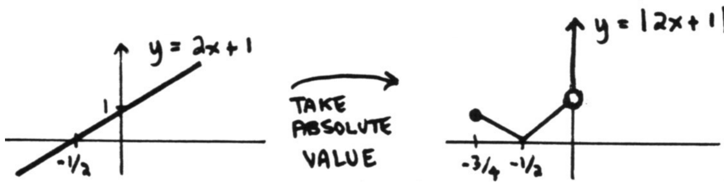
$$f(x) = |2x+1| = |2(x+\frac{1}{2})| = 2|x+\frac{1}{2}|$$

Then, the graph of f is found by a series of transformations:



The minimum value of f on I is 0; the maximum value is 5. The only minimum point is $(-\frac{1}{2}, 0)$; the only maximum point is $(2, 5)$.

8. Here's a second way to graph the function $f(x) = |2x+1|$. First, graph the line $y = 2x+1$. Then, 'flip up' the part of the line that has negative y-values:



The minimum value is 0; there is no maximum value. The only minimum point is $(-\frac{1}{2}, 0)$.

9. TRUE. This is a consequence of the Max-Min Theorem.

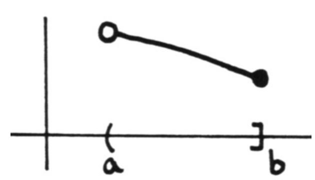
   Contrapositive: If f does not attain a maximum value on $[a, b]$, then f is not continuous on $[a, b]$.

10. TRUE. This is a consequence of the Max-Min theorem.

    Contrapositive: If f is continuous on $[a, b]$, then f attains a maximum value on $[a, b]$.

11. FALSE. There are functions that are continuous on $(a, b]$, but do not attain a maximum value on $[a, b]$. (You have probably guessed what it means to be 'continuous on $(a, b]$'. It means that f is continuous on (a, b), and well-behaved at the right-hand endpoint.)

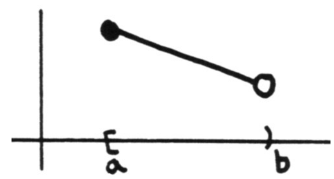
    Counterexample: Let f be the function graphed below. Then, the hypothesis 'f is continuous on $(a, b]$' is TRUE, but the conclusion 'f attains a maximum value on $(a, b]$' is FALSE.



Contrapositive: If f does not attain a maximum value on $(a, b]$, then f is not continuous on $(a, b]$.

12. FALSE. There are functions that are continuous on $[a, b)$, but do not attain a minimum value on $[a, b]$. (You have probably guessed what it means to be 'continuous on $[a, b)$'. It means that f is continuous on (a, b), and well-behaved at the left-hand endpoint.)

    Counterexample: Let f be the function graphed below. Then, the hypothesis 'f is continuous on $[a, b)$' is TRUE, but the conclusion 'f attains a minimum value on $[a, b)$' is FALSE.

    Contrapositive: If f does not attain a minimum value on $[a, b)$, then f is not continuous on $[a, b)$.

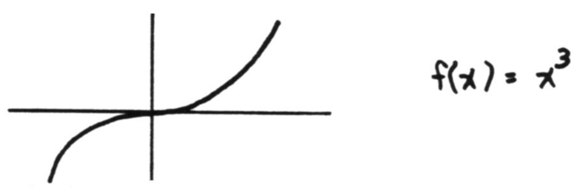
13. TRUE. Whenever f is continuous on $(0, 5)$, then it is also continuous on the closed interval $[1, 2]$. Thus, by the Max-Min theorem, f attains both a maximum and minimum value on $[1, 2]$.

    Contrapositive: If f does NOT attain both a maximum and minimum value on $[1, 2]$, then f is not continuous on $(0, 5)$.

14. TRUE. See (13).

15. FALSE

    Counterexample: Let f be the function graphed below. Then the hypothesis 'f is continuous on $\mathbb{R}$' is TRUE, but the conclusion 'f attains a maximum value on $\mathbb{R}$' is FALSE.

    Contrapositive: If f does not attain a maximum value on $\mathbb{R}$, then f is not continuous on $\mathbb{R}$.

16. FALSE. See (15).