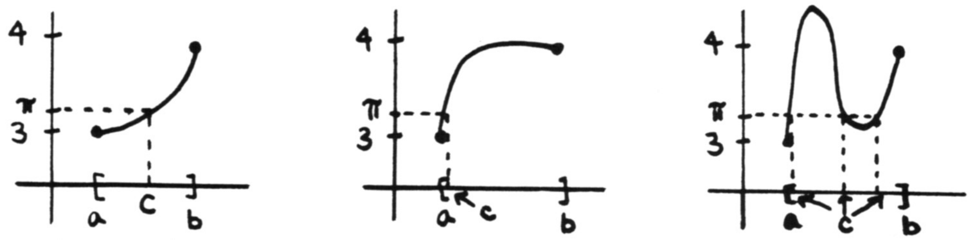
SECTION 3.6 The Intermediate Value Theorem

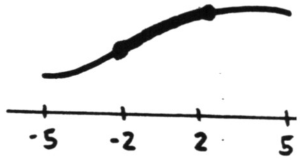
IN-SECTION EXERCISES:

EXERCISE 1.

1. There are many correct graphs possible. A few are shown below. Since f is continuous on $[a,b]$ and π is between $f(a)=3$ and $f(b)=4$, there MUST (by the Intermediate Value Theorem) be a number $c \in (a,b)$ with $f(c)=\pi$. Such value(s) of c are shown on the graphs below.



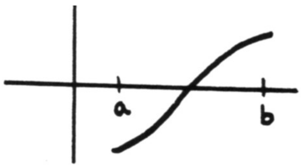
2. If f is continuous on $(-5,5)$, then for every $c \in (-5,5)$, $\lim_{x\to c} f(x) = f(c)$. In particular, when x is 2 or -2,
$$\lim_{x\to -2} f(x) = f(-2) \quad \text{and} \quad \lim_{x\to 2} f(x) = f(2)\ .$$
Since the two-sided limits exist, so do all one-sided limits; in particular,



$$\lim_{x\to -2^+} f(x) = f(-2) \quad \text{and} \quad \lim_{x\to 2^-} f(x) = f(2)\ .$$
Thus, f is continuous on $[-2,2]$.

Indeed, if f is continuous on the interval (a,b), then it is continuous on *any* closed interval contained in (a,b).

3. Since f is continuous on $[a,b]$, and 0 is a number between $f(a)$ and $f(b)$, the Intermediate Value Theorem guarantees existence of $c \in (a,b)$ with $f(c)=0$.



4. If f were continuous on $[0,1]$, then there would have to be a number c between 0 and 1 with $f(c)=2.5$. Thus, f is NOT continuous on $[0,1]$.

EXERCISE 2.

1.

SOLN IN (1.7, 1.8)

x	$x^4 - 8x^2$
0.8000	-4.7104
0.9000	-5.8239
1.0000	-7.0000
1.1000	-8.2159
1.2000	-9.4464
1.3000	-10.6639
1.4000	-11.8384
1.5000	-12.9375
1.6000	-13.9264
1.7000	-14.7679
1.8000	-15.4224
1.9000	-15.8479
2.0000	-16.0000

SOLN IN (1.73, 1.74)

x	$x^4 - 8x^2$
1.7000	-14.7679
1.7100	-14.8424
1.7200	-14.9151
1.7300	-14.9857
1.7400	-15.0544
1.7500	-15.1211
1.7600	-15.1857
1.7700	-15.2481
1.7800	-15.3084
1.7900	-15.3665
1.8000	-15.4224

2.

SOLN IN (2.2, 2.3)

x	$x^4 - 8x^2$
2.0000	-16.0000
2.1000	-15.8319
2.2000	-15.2944
2.3000	-14.3359
2.4000	-12.9024
2.5000	-10.9375
2.6000	-8.3824
2.7000	-5.1759
2.8000	-1.2544
2.9000	3.4481
3.0000	9.0000

3. Exact solutions:

$$\begin{aligned}
x^4 - 8x^2 = -15 \quad &\iff \quad x^4 - 8x^2 + 15 = 0 \\
&\iff \quad (x^2)^2 - 8(x^2) + 15 = 0 \\
&\iff \quad (x^2 - 5)(x^2 - 3) = 0 \\
&\iff \quad (x - \sqrt{5})(x + \sqrt{5})(x - \sqrt{3})(x + \sqrt{3}) = 0 \\
&\iff \quad x = \pm\sqrt{5} \ \text{ or } \ x = \pm\sqrt{3}\ .
\end{aligned}$$

In Exercise 1, you were finding the solution $x = \sqrt{3}$. In Exercise 2, you were finding the solution $x = \sqrt{5}$.

EXERCISE 3.
First,

$$x^3 - x^2 = 5x - 5 \quad \iff \quad x^3 - x^2 - 5x + 5 = 0\ .$$

Define $f(x) := x^3 - x^2 - 5x + 5$. The tables of function values shown below help us to locate a solution:

SOLN IN (2.2, 2.3)

x	$f(x)$
2.0000	-1.0000
2.1000	-0.6490
2.2000	-0.1920
2.3000	0.3770
2.4000	1.0640
2.5000	1.8750
2.6000	2.8160
2.7000	3.8930
2.8000	5.1120
2.9000	6.4790
3.0000	8.0000

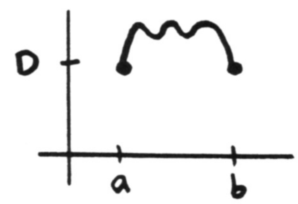
SOLN IN (2.23, 2.24)

x	$f(x)$
2.2000	-0.1920
2.2100	-0.1402
2.2200	-0.0874
2.2300	-0.0333
2.2400	0.0218
2.2500	0.0781
2.2600	0.1356
2.2700	0.1942

EXERCISE 4.

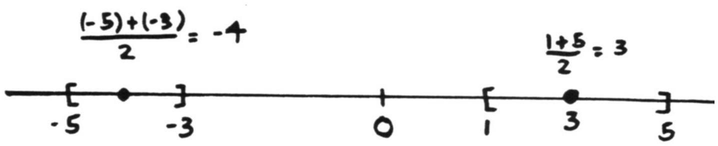
1. This is a bit of a trick question. Since we are told that $f(a) = f(b) = D$, take $d = a$ or $d = b$. Then, $f(d) = D$.

2. There need *not* be any $d \in (a, b)$ with $f(d) = D$, as the sketches below illustrate.



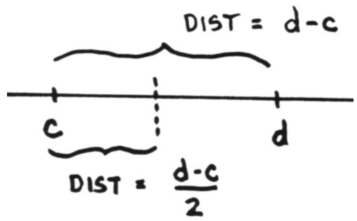
EXERCISE 5.

1.



2. Let $c < d$. Then, the distance from c to d is $d - c$. The number exactly halfway between c and d is:

$$c + \frac{d-c}{2} = \frac{2c}{2} + \frac{d-c}{2} = \frac{2c+d-c}{2} = \frac{c+d}{2} \ .$$



3. There are really two cases to be considered:

   Case 1. $f(a) = f(b)$

   In this case,

$$\frac{f(a)+f(b)}{2} = \frac{f(a)+f(a)}{2} = f(a) \ ,$$

   and this value is taken on when $x = a$, or when $x = b$.

   Case 2. $f(a) \neq f(b)$

   In this case, the average value $\frac{f(a)+f(b)}{2}$ is a number between $f(a)$ and $f(b)$, and the Intermediate Value Theorem says that there must exist $d \in (a, b)$ with $f(d) = \frac{f(a)+f(b)}{2}$.

EXERCISE 6.

The *hypotheses* of a theorem are the things that MUST be true, in order to use the theorem. That is, the conclusion of the theorem is only guaranteed to be true if the hypotheses are true. If the hypotheses are not true, the conclusion of the theorem *may* be true or *may not* be true; further investigation is required.

The *hypothesis* of a (true) implication must be true to guarantee that the conclusion of the implication is true. That is, if '$A \implies B$' is a true sentence, this means that whenever the hypothesis A is true, then the conclusion B must also be true. If A isn't true, no information can be reached about B.

Therefore, the words have very similar meanings in both contexts.

EXERCISE 7.

1. TRUE. Whenever the sentence $x = 3$ is true, so is the sentence $x^2 = 9$. Indeed, the hypothesis is only true when x is 3; and $3^2 = 9$.

2. FALSE. Just because x is a number which, when squared, equals 9, we cannot conclude that x is 3.

   Here's a counterexample:

   Let $x = -3$. Then the hypothesis $(-3)^2 = 9$ is true, but the conclusion $-3 = 3$ is false.

   This is the only possible counterexample for this problem.
3. TRUE. Whenever x is equal to 2, the absolute value of x is also equal to 2.
4. FALSE. Just because x is a number whose distance from zero is 2, we cannot conclude that x is 2.

   Here's a counterexample:

   Let $x = -2$. Then the hypothesis $|-2| = 2$ is true, but the conclusion $-2 = 2$ is false.

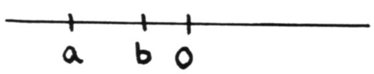
   This is the only possible counterexample for this problem.
5. You must think to yourself: whenever a is any number that lies to the left of b on the number line, are we guaranteed that a's distance from zero is less than b's distance from zero?

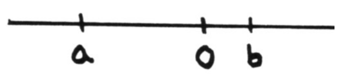
   Several sketches show that the answer is NO. That is, the sentence is FALSE. Here's a counterexample:

   Let $a = -3$ and $b = -1$. Then, the hypothesis $-3 < -1$ is true, but the conclusion $|-3| < |-1|$ is false.

   There are an infinite number of counterexamples for this problem. Here's another:

   Let $a = -4$ and $b = 3$. Then, the hypothesis $-4 < 3$ is true, but the conclusion $|-4| < |3|$ is false.





6. TRUE. Whenever a and b are both positive numbers with $a < b$, then a's distance from 0 is less than b's distance from zero.

EXERCISE 8.
Keep in mind that we are assuming that the sentence $A \implies B$ is TRUE. There are three lines of the truth table where this happens:

A	B	A ⇒ B
T	T	T
T	F	F
F	T	T
F	F	T

1. Nothing can be concluded about the truth value of A. It could be true (line 1) or false (lines 3 and 4).
2. Nothing can be concluded about the truth value of B. It could be true (lines 1 and 3) or false (line 4).
3. If A is true, then B must be true (line 1).
4. If B is true, nothing can be concluded about the truth value of A. A might be true (line 1) or false (line 3).
5. If A is false, nothing can be concluded about the truth value of B. B might be true (line 3) or false (line 4).
6. If B is false, then A must be false (line 4). For if A were true, then B would also have to be true!

END-OF-SECTION EXERCISES:
1. TRUE. This is a statement of the Intermediate Value Theorem.
2. TRUE. This is an application of the Intermediate Value Theorem.
3. TRUE. Whenever the hypothesis of an implication is false, the implication itself is automatically true. (In this situation, the implication is said to be *vacuously true*).

4. FALSE. Look at the truth table for $A \implies B$. There are two lines where B is false, line 2 and line 4. Is the sentence false in both cases? No. Here's a counterexample:

   Let A be false and B be false. Then, the hypothesis 'B is false' is true, but the conclusion 'the sentence $A \implies B$ is false' is false.
5. TRUE. Look at the truth table for $A \implies B$. There are two lines where B is true; line 1 and line 3. In both cases, the sentence $A \implies B$ is true. That is, whenever the conclusion of an implication is true, the implication itself is true.
6. FALSE. Look at the truth table for $A \implies B$. There are two lines where A is true; line 1 and line 2. Is the sentence $A \implies B$ true in both cases? No. Here's a counterexample:

   Let A be true and B be false. Then the hypothesis 'A is true' is true, but the conclusion '$A \implies B$ is true' is false.
7. TRUE. Whenever t is a number whose distance from 0 is 0, t itself must be zero.
8. FALSE. Just because t is a number whose distance from 0 is 1, we cannot conclude that t is 1. Here's a counterexample:

   Let $t = -1$. Then the hypothesis $|-1| = 1$ is true, but the conclusion $-1 = 1$ is false.
9. TRUE. Whenever t is 1, it is also true that $|t|$ is 1.
10. TRUE. Whenever t is -1, it is also true that $|t|$ is 1.