SECTION 3.5 Indeterminate Forms

IN-SECTION EXERCISES:

EXERCISE 1.

$$\begin{aligned}\lim_{x\to 1}\frac{3-3x}{x-1} &= \lim_{x\to 1}\frac{-3(x-1)}{x-1}\\ &= \lim_{x\to 1}(-3)\\ &= -3\end{aligned}$$

EXERCISE 2.

1. The sentence '$f = g$' is being defined. The definition tells us what it means for *functions to be equal.*
2. The symbol ' $\iff$ ' means that the two component sentences being compared always have the same truth values. They are interchangeable.
3. If $g = h$, then we know that $\mathcal{D}(g) = \mathcal{D}(h)$, and $g(x) = h(x)$ for all x in the common domain.
4. You can conclude that $g = h$.
5. The functions f and g are very similar, but they are not identical. By the domain convention, the domain of f is all real numbers except 1. The domain of g is $\mathbb{R}$. Since the functions have different domains, they are not equal.
6. The functions f and g both have the same domain: all real numbers except 0. And, for $x \neq 0$,

$$\frac{3x}{x^2} = \frac{3}{x}\ .$$

Thus, the functions f and g are equal, i.e., $f = g$.

1.

EXERCISE 3.

$$\begin{aligned}\lim_{x\to 1}\frac{x^2+x-2}{3x^2-x-2} &= \lim_{x\to 1}\frac{(x+2)(x-1)}{(x-1)(3x+2)}\\ &= \lim_{x\to 1}\frac{x+2}{3x+2}\\ &= \frac{1+2}{3(1)+2}\\ &= \frac{3}{5}\end{aligned}$$

2.

$$\begin{aligned}\lim_{x\to -2}\frac{x^2+x-2}{x^3-7x-6} &= \lim_{x\to -2}\frac{(x+2)(x-1)}{(x+2)(x^2-2x-3)}\\ &= \lim_{x\to -2}\frac{x-1}{x^2-2x-3}\\ &= \frac{-2-1}{(-2)^2-2(-2)-3}\\ &= -\frac{3}{5}\ .\end{aligned}$$

$$\begin{array}{r|l} & x^2 - 2x - 3 \\ \hline x+2 & x^3 - 7x - 6 \\ & -(x^3 + 2x^2) \\ \hline & -2x^2 - 7x - 6 \\ & -(-2x^2 - 4x) \\ \hline & -3x - 6 \\ & -3x - 6 \\ \hline & 0 \end{array}$$

EXERCISE 4.

1.

$$\lim_{x\to 2} \frac{e^x(x-2)}{2-x} = \lim_{x\to 2} \frac{-e^x(2-x)}{2-x} = -\lim_{x\to 2} e^x = e^2$$

2.

$$\frac{xe^x - 2e^x}{2-x} = \frac{e^x(x-2)}{2-x} = \frac{-e^x(2-x)}{2-x} \overset{x\neq 2}{=} -e^x$$

EXERCISE 5.

1. Direct substitution yields a $\frac{0}{0}$ situation. Since 1 is a zero of the numerator, $x-1$ is a factor. Long division yields:

$$
\begin{array}{r}
x^2 + x + 1 \\
x-1\,\overline{)\,x^3 - 1} \\
-(x^3 - x^2) \\
\hline
x^2 - 1 \\
-(x^2 - x) \\
\hline
x - 1 \\
\underline{x - 1}
\end{array}
$$

Then,

$$
\begin{aligned}
\lim_{x\to 1} \frac{x^3-1}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1} \\
&= \lim_{x\to 1}(x^2+x+1) \\
&= 1^2+1+1 = 3\ .
\end{aligned}
$$

2. Direct substitution yields a $\frac{0}{0}$ situation. Since 2 is a zero of the numerator, $x-2$ is a factor. Long division yields:

$$
\begin{array}{r}
x^2 - 2x - 15 \\
x-2\,\overline{)\,x^3 - 4x^2 - 11x + 30} \\
-(x^3 - 2x^2) \\
\hline
-2x^2 - 11x + 30 \\
-(-2x^2 + 4x) \\
\hline
-15x + 30 \\
\underline{-15x + 30}
\end{array}
$$

Then,

$$
\begin{aligned}
\lim_{x\to 2} \frac{x^3-4x^2-11x+30}{x-2} &= \lim_{x\to 2} \frac{(x-2)(x^2-2x-15)}{x-2} \\
&= \lim_{x\to 2}(x^2-2x-15) \\
&= 2^2-2(2)-15 = -15\ .
\end{aligned}
$$

EXERCISE 6.

1. The sentence
$$\frac{(x+1)-1}{x(\sqrt{x+1}+1)} = \frac{1}{\sqrt{x+1}+1}$$
is true for *most* values of x; but it is not true when $x=0$. When $x=0$, the expression on the left of the '=' sign is undefined, but the expression on the right is defined, and equals $\frac{1}{\sqrt{0+1}+1} = \frac{1}{2}$. Attention is drawn to this slight difference in the expressions by using a 'restricted equal sign',
$$\frac{(x+1)-1}{x(\sqrt{x+1}+1)} \overset{\text{for } x\neq 0}{=} \frac{1}{\sqrt{x+1}+1} \ .$$

2. The sentence
$$\lim_{x\to 0} \frac{(x+1)-1}{x(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1}$$
is true. The sentence states that two *numbers* are equal. When evaluating a limit $\lim_{x\to c} f(x)$, the function f may be replaced by *any* function that agrees with it, except possibly at c. Here, $c=0$, and the two functions $\frac{(x+1)-1}{x(\sqrt{x+1}+1)}$ and $\frac{1}{\sqrt{x+1}+1}$ agree everywhere except at 0.

EXERCISE 7.

Direct substitution yields a $\frac{0}{0}$ situation. In order to get the function in a better form to see what's happening *near* $x=0$, we rationalize the denominator:
$$\begin{aligned}\frac{3x}{\sqrt{x+4}-2} &= \frac{3x}{\sqrt{x+4}-2}\cdot\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\\ &= \frac{3x(\sqrt{x+4}+2)}{(x+4)-4}\\ &\overset{\text{for } x\neq 0}{=} 3(\sqrt{x+4}+2)\ .\end{aligned}$$

Then,
$$\begin{aligned}\lim_{x\to 0}\frac{3x}{\sqrt{x+4}-2} &= \lim_{x\to 0} 3(\sqrt{x+4}+2)\\ &= 3(\sqrt{0+4}+2) = 3(2+2) = 12\ .\end{aligned}$$

END-OF-SECTION EXERCISES:

1. SEN; FALSE. The sentence $\frac{x^3-1}{x-1} = x^2+x+1$ is NOT true for ALL real numbers x. (The expression on the right was obtained by long division.) When $x=1$, the expression on the left is undefined , but the expression on the right equals 3.
2. SEN; TRUE.
3. SEN; FALSE. The functions f and g have different domains, hence are different functions. The domain of f is all real numbers except 1. The domain of g is $\mathbb{R}$.
4. EXP
5. SEN; TRUE. The function $\frac{x^3-1}{x-1}$ has been replaced by a function that agrees with it everywhere except at $x=1$.
6. SEN; TRUE
7. SEN; TRUE. (Either both limits do not exist; or they both exist, and are equal.)
8. SEN; CONDITIONAL. The truth of this sentence depends upon the choices made for the functions f and g.

9. Direct substitution yields a $\frac{0}{0}$ situation. Since -1 is a zero of the numerator, $x-(-1)=x+1$ is a factor. Either use long division, or factor by grouping:

$$\begin{aligned} x^3+x^2-3x-3 &= x^2(x+1)-3(x+1) \\ &= (x^2-3)(x+1)\ . \end{aligned}$$

Then,

$$\begin{aligned} \lim_{x\to -1}\frac{x^3+x^2-3x-3}{x+1} &= \lim_{x\to -1}\frac{(x+1)(x^2-3)}{x+1} \\ &= \lim_{x\to -1} x^2-3 \\ &= (-1)^2-3=-2\ . \end{aligned}$$

10. The function in the limit is continuous at $x=1$, so evaluation of the limit is as easy as direct substitution:

$$\lim_{x\to 1}\frac{x^3+x^2-3x-3}{x+1}=\frac{1^3+1^2-3(1)-3}{1+1}=\frac{-4}{2}=-2\ .$$

11. The function in the limit is continuous at $x=2$:

$$\lim_{x\to 2}\frac{x+2}{x^2+4x+4}=\frac{2+2}{2^2+4(2)+4}=\frac{4}{16}=\frac{1}{4}\ .$$

12. Direct substitution yields a $\frac{0}{0}$ situation.

$$\begin{aligned} \lim_{x\to -2}\frac{x+2}{x^2+4x+4} &= \lim_{x\to -2}\frac{x+2}{(x+2)^2} \\ &= \lim_{x\to -2}\frac{1}{x+2}\ ; \end{aligned}$$

since $\lim_{x\to -2}\frac{1}{x+2}$ does not exist, neither does the original limit.

13. In the text, it was shown that

$$\lim_{x\to 0^+}(1+x)^{1/x}$$

equals the irrational number e.

Thus, using the dummy variable t,

$$\lim_{t\to 0^+}(1+t)^{1/t}=e\ .$$

14. Using the dummy variable y,

$$\lim_{y\to 0^+}(1+y)^{1/y}=e\ .$$