SECTION 3.3 Properties of Limits

IN-SECTION EXERCISES:

EXERCISE 1.

In the equation $ax + b = c$, mathematical conventions dictate that a, b and c are constants, and x is the variable.

Uniqueness of Solutions: Suppose that both X and Y are solutions of $ax + b = c$, where $a \neq 0$.

Since X is a solution, $aX + b = c$.

Since Y is a solution, $aY + b = c$.

Thus, $aX + b = aY + b$ is true (since both numbers equal c). But,

$$\begin{aligned} aX + b = aY + b \quad &\iff \quad aX = aY \quad \text{(subtract } b\text{)} \\ &\iff \quad X = Y \quad \text{(divide by } a \neq 0\text{)} \end{aligned}$$

Since the sentence $aX + b = aY + b$ was true, so is the sentence $X = Y$.

EXERCISE 2.

The author's goal was to show that a number cannot be in two disjoint intervals at the same time. If ϵ represents the distance between l and k, then the intervals $(l - \frac{\epsilon}{2}, l + \frac{\epsilon}{2})$ and $(k - \frac{\epsilon}{2}, k + \frac{\epsilon}{2})$ are disjoint; that is, they do not overlap at all. Thus, $\frac{\epsilon}{2}$ would have worked.

The intervals $(l - \frac{\epsilon}{4}, l + \frac{\epsilon}{4})$ and $(k - \frac{\epsilon}{4}, l + \frac{\epsilon}{4})$ are clearly disjoint; so $\frac{\epsilon}{4}$ would also have worked.

The author chose $\frac{\epsilon}{3}$, because the disjointness of the intervals is clear (there is $\frac{\epsilon}{3}$ of 'space' between them), and 3 is the smallest integer denominator that yields a clear separation.

EXERCISE 3.

Here's a statement of the Uniqueness of Limits Theorem from **Calculus, One and Several Variables**, fourth edition, S.L. Salas and Einar Hille, 1982, p. 56.

THEOREM. If

$$\lim_{x \to c} f(x) = l \quad \text{and} \quad \lim_{x \to c} f(x) = m \ ,$$

then

$$l = m \ .$$

The statement of the theorem is almost identical. In the Fisher text, the words 'suppose that' were used instead of 'if'. Also, the letters l and k were used, instead of l and m.

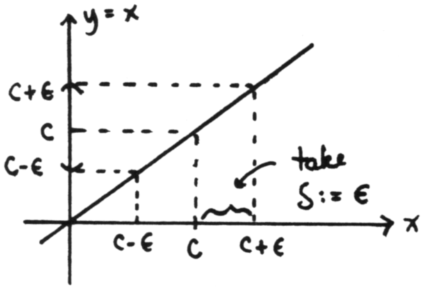
However, the proof in Salas & Hille is dramatically different. The authors chose to prove the result by letting ϵ denote any positive number (arbitrarily small), and showing that $|l - m| < \epsilon$. Thus, the distance from l to m must be strictly less than *every* positive number. Thus, l must equal m.

EXERCISE 4.

In the previous proof, δ can be chosen to be *any* positive number. The author just chose 1, because it's simple.

EXERCISE 5.

1. The property $\lim_{x \to c} x = c$ tells you that it is easy to evaluate the limit of $f(x) = x$ as x approaches c; just evaluate f at c!
2. The sketch in (3) certainly convinces the author that this property is true. To get $f(x) = x$ within ϵ of c, it is only necessary to keep x within ϵ of c! So, choose $\delta := \epsilon$.
3. The 4-step process is summarized in the sketch at right.



EXERCISE 6.

1. Properties (O2) and (O1) can be used to conclude that the limit of a difference is the difference of the limits:

$$\begin{aligned}
\lim_{x\to c}[f(x) - g(x)] &= \lim_{x\to c}[f(x) + (-g(x))] && \text{(rewrite)}\\
&= \lim_{x\to c} f(x) + \lim_{x\to c}(-g(x)) && \text{(O2)}\\
&= \lim_{x\to c} f(x) + (-1)\lim_{x\to c} g(x) && \text{(O1)}\\
&= \lim_{x\to c} f(x) - \lim_{x\to c} g(x) && \text{(rewrite)}
\end{aligned}$$

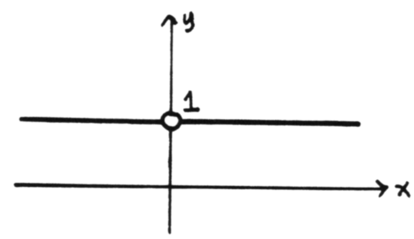
   The fact that both individual limits exist was used *repeatedly* in this argument. For example, since $\lim_{x\to c} g(x)$ exists, so does $\lim_{x\to c} -g(x)$; this allowed us to break up the sum in the first line above.

2. Every '=' sign works in TWO directions! If $a = b$, then a is equal to b, and b is equal to a. Thus, property (O2) can certainly be used 'backwards', whenever it is convenient to do so!

EXERCISE 7.

1. Whenever $x \ne 0$, $x \cdot \frac{1}{x} = 1$. Thus, whenever x is near 0, $x \cdot \frac{1}{x}$ is near 1. See the sketch below. Thus,

$$\lim_{x\to 0} x \cdot \frac{1}{x} = 1\ .$$

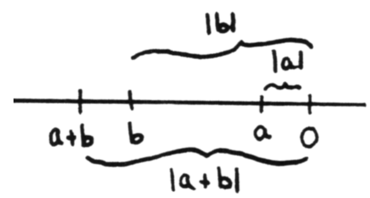


2. The student has not met the hypotheses of the theorem regarding operations with limits. In order to write the limit of a product as the product of a limit, it must be known that *each individual limit exists.* In this case, the limit $\lim_{x\to 0} \frac{1}{x}$ does not exist, so the theorem cannot be used.

EXERCISE 8.

1. Suppose that both a and b are negative. Then, $|a| = -a$ and $|b| = -b$. Also, since both a and b are negative, so is $a + b$, so that $|a + b| = -(a + b)$. Thus,

$$\begin{aligned}
|a + b| &= -(a + b)\\
&= (-a) + (-b)\\
&= |a| + |b|\ ,
\end{aligned}$$



   and we actually have equality in this case.

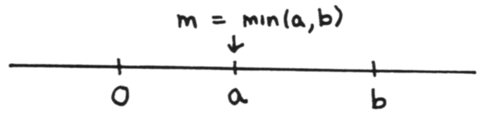
2. The case '$a < 0$ and $b \ge 0$' represents the case where one number is negative, and one number is nonnegative. The case '$b < 0$ and $a \ge 0$' describes precisely the same situation! Thus, these cases are no different. In other words, a renaming of the variables (rename 'a' as 'b', and 'b' as 'a') yields precisely the same situation.

EXERCISE 9.

$$\begin{aligned}
|f(x) + g(x) - (l + k)| &= |(f(x) - l) + (g(x) - k)| && \text{Reason: regroup}\\
&\le |f(x) - l| + |g(x) - k| && \text{Reason: triangle inequality}\\
&< \epsilon/2 + \epsilon/2 && \text{Reason: See (*) below.}\\
&= \epsilon && \text{Reason: addition}
\end{aligned}$$

(*) By assumption, x is in the domain of both f and g, and is within δ of c, where δ is the minimum of δ_1 and δ_2.
Since x is within δ_1 of c, $|f(x) - l| < \frac{\epsilon}{2}$.
Since x is within δ_2 of c, $|g(x) - k| < \frac{\epsilon}{2}$.
Thus, the sum is less than $\frac{\epsilon}{2} + \frac{\epsilon}{2}$.

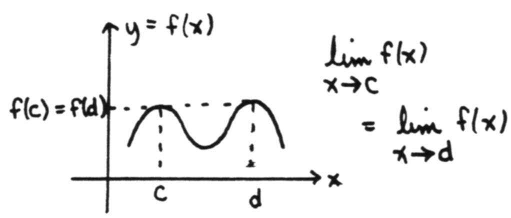
EXERCISE 10.
The sketch below certainly convinces the author that if $m :=$ minimum(a, b), then both $m \le a$ and $m \le b$.

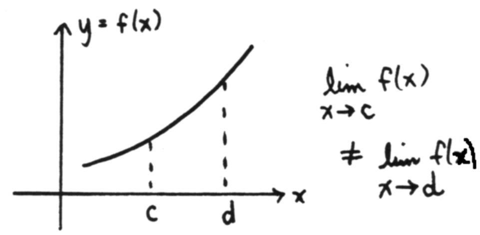


EXERCISE 11.
Assuming that all individual limits exist,

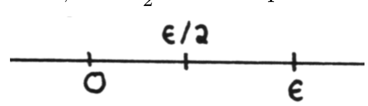
$$\begin{aligned}
\lim_{x\to c} f(x)g(x)h(x) &= \lim_{x\to c}[f(x)g(x)]h(x) && \text{(regroup)}\\
&= \lim_{x\to c}[f(x)g(x)] \lim_{x\to c} h(x) && \text{(use (O3) once)}\\
&= \lim_{x\to c} f(x) \lim_{x\to c} g(x) \lim_{x\to c} h(x)\ . && \text{(use (O3) again)}
\end{aligned}$$

END-OF-SECTION EXERCISES:
1. SEN; TRUE. This is a statement of the Uniqueness of Limits Theorem. The dummy variable 'y' was used instead of 'x' in the second limit, to represent a typical input that is approaching c.
2. SEN; TRUE. This is a statement of the Uniqueness of Limits Theorem.
3. SEN; TRUE. Only the dummy variable has been changed. For a given function f and a given number c, either both limits will not exist, or they will both exist and be equal.
4. SEN; CONDITIONAL. This sentence says that whenever the inputs to f approach both c and d, the function values approach the same number. The sketches below show a case where the sentence is true, and a case where it is false.





5. SEN; TRUE. If ϵ is any positive number, then $\frac{\epsilon}{2}$ is also a positive number.



6. SEN; TRUE. If $\frac{\epsilon}{2}$ is a positive number, then ϵ must also be a positive number.
7. SEN; TRUE. The two sentences being compared always have the same truth values, regardless of the number chosen for ϵ. (What happens if $\epsilon = -1$?)
8. SEN; TRUE. Multiplying an inequality by a positive number always yields an equivalent inequality in the same direction. Thus, the two sentences being compared always have the same truth values, regardless of the number chosen for ϵ. (What happens if $\epsilon = -1$?)
9. SEN; FALSE. The two sentences being compared do NOT always have the same truth values. Choose, say, $\epsilon = .05$. Then the first sentence '$.05 > 0$' is true, but the second sentence '$(.05 - .1) > 0$' is false. Thus, the two sentences cannot be used interchangeably.

10. SEN; TRUE. This is property (P1).
11. SEN; TRUE. This is an application of property (P3).
12. SEN; TRUE. This is an application of property (P2).
13. SEN; CONDITIONAL. (Careful!) If both individual limits exist, then this sentence is true. However, it may not be true if one of the individual limits fails to exist.
14. SEN; TRUE. This is operation (O2).
15.

$$\begin{aligned}\lim_{t\to c}[f(t)+g(t)] &= \lim_{t\to c} f(t) + \lim_{t\to c} g(t)\\ &= (-1)+2\\ &= 1 \ .\end{aligned}$$

16.

$$\begin{aligned}\lim_{t\to c}(f-g)(t) &= \lim_{t\to c} f(t) - g(t)\\ &= \lim_{t\to c} f(t) - \lim_{t\to c} g(t)\\ &= (-1)-2\\ &= -3 \ .\end{aligned}$$

17. There is not enough information to evaluate this limit. We don't know anything about the behavior of f and g, as the inputs approach the number d.
18. Since all the individual limits exist, repeated application of the properties of limits yields:

$$\lim_{x\to c}[3g(x)-f(x)]\cdot h(x) = [3(2)-(-1)]\cdot 0 = 0 \ .$$