SECTION 2.4 One-to-One Functions and Inverse Functions

IN-SECTION EXERCISES:

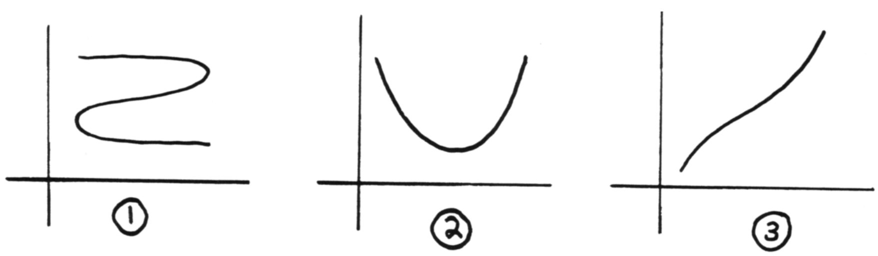
EXERCISE 1.

$\forall\, y \in \mathcal{R}(f),\ \ \exists\,!\ x \in \mathcal{D}(f)$ (In text, this sentence should begin with the *words* ‘For all’, not the symbol ‘$\forall$’.)
This is the ‘1-1’ condition for a function f.

EXERCISE 2.

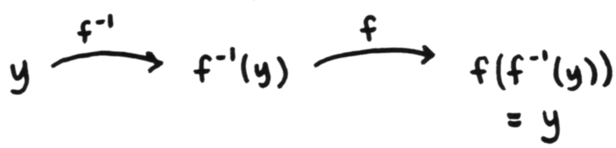
There are many correct examples possible for each of these questions. See the graphs below.

1. The graph of a non-function will not pass a vertical line test.
2. The graph must pass a vertical line test, but not a horizontal line test.
3. The graph must pass both a vertical and horizontal line test.
4. The fact that ‘y is a function of x’ says that the graph must pass a vertical line test. The fact that ‘x is a function of y’ says that the graph must pass a horizontal line test. Use the graph from (3).
5. Here, the graph must pass a vertical line test, but not a horizontal line test. Use the graph from (2).
6. Every 1-1 *function* is firstly a *function*. Thus, it is not possible to be a 1-1 function without being a function.



EXERCISE 3.

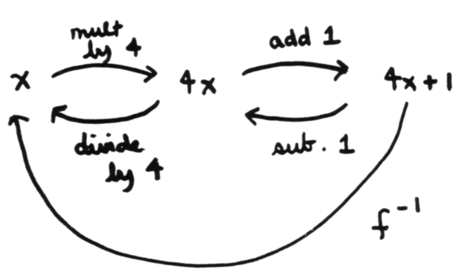
Given an element $y \in \mathcal{R}(f)$, the function f^{-1} takes y to $f^{-1}(y) \in \mathcal{D}(f)$. Then, the function f takes $f^{-1}(y)$ back to y, so that $f(f^{-1}(y)) = y$. The dummy variable y was chosen, since here we were beginning the process with an element in the range, and the letter ‘y’ is typically used for range elements.



EXERCISE 4.

The two functions in this exercise are lines. All non-vertical, non-horizontal lines are 1-1.

1. Mapping approach: The function $f(x) = 4x + 1$ takes an input x, multiplies it by 4, then adds 1. To ‘undo’ this, f^{-1} must take an input x, subtract 1, then divide by 4. Thus, $f^{-1}(x) = \frac{x-1}{4}$.



Algebraic approach: Treat $f^{-1}(x)$ as the 'unknown', and solve for it:

$$\begin{aligned} f(f^{-1}(x)) &= x \\ 4(f^{-1}(x)) + 1 &= x \\ f^{-1}(x) &= \frac{x-1}{4} \end{aligned}$$

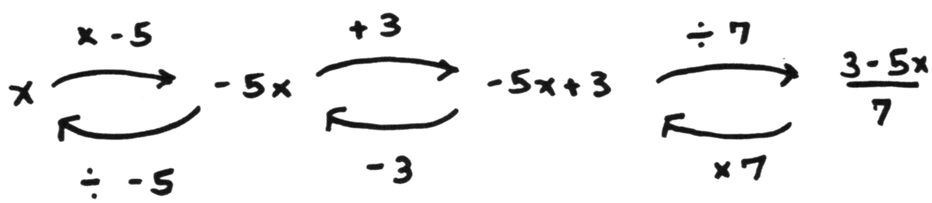
Checking both conditions:

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(4x+1) \\ &= \frac{(4x+1)-1}{4} \\ &= x \end{aligned}$$

and

$$\begin{aligned} f(f^{-1}(x)) &= f(\frac{x-1}{4}) \\ &= 4(\frac{x-1}{4}) + 1 \\ &= x \end{aligned}$$

2. Mapping approach: The function $g(x) = \frac{3-5x}{7}$ takes an input x, multiplies by -5, adds 3, then divides by 7. To 'undo' this, g^{-1} must take an input x, multiply by 7, subtract 3, then divide by -5. Thus, $g^{-1}(x) = \frac{7x-3}{-5}$.



Algebraic approach: Treat $g^{-1}(x)$ as the 'unknown', and solve for it:

$$\begin{aligned} g(g^{-1}(x)) &= x \\ \frac{3-5g^{-1}(x)}{7} &= x \\ g^{-1}(x) &= \frac{7x-3}{-5} \end{aligned}$$

Checking both conditions:

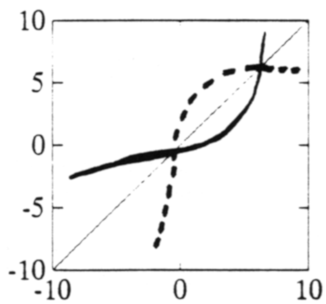
$$
\begin{aligned}
g^{-1}(g(x)) &= g^{-1}(\frac{3-5x}{7}) \\
&= \frac{7\frac{3-5x}{7}-3}{-5} \\
&= \frac{3-5x-3}{-5} \\
&= x
\end{aligned}
$$

and

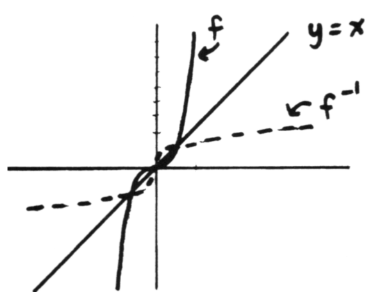
$$
\begin{aligned}
g(g^{-1}(x)) &= g(\frac{7x-3}{-5}) \\
&= \frac{3-5\frac{7x-3}{-5}}{7} \\
&= \frac{3+7x-3}{7} \\
&= x
\end{aligned}
$$

EXERCISE 5.

1.



2. A quick sketch of $f(x) = x^3$ shows that it is 1-1. To 'undo' the cubing function, take the cube root. Thus, $f^{-1}(x) = \sqrt[3]{x}$. Both functions are graphed below.



EXERCISE 6.

1. The domain of the natural logarithm function is the same as the range of the exponential function. Thus, the domain of the natural logarithm function is $(0, \infty)$.
2. The range of the natural logarithm function is the same as the domain of the exponential function. Thus, the range of the natural logarithm function is $\mathbb{R}$.
3. As x approaches infinity, so does $\ln x$. However, $\ln x$ approaches infinity *much slower* than e^x approaches infinity.
4. As x gets closer and closer to zero (from the right), $\ln x$ approaches negative infinity.

EXERCISE 7.

1. Just make the appropriate substitutions.
2. Just make the appropriate substitutions.

EXERCISE 8.

After one quarter: $2000 + (2000)(0.10)(0.25) = 2050.00$

After two quarters: $2050 + (2050)(0.10)(0.25) = 2050.00 + 51.25 = 2101.25$

After three quarters: $2101.25 + (2101.25)(0.10)(0.25) = 2101.25 + 52.53 = 2153.78$

After four quarters: $2153.78 + (2153.78)(0.10)(0.25) = 2153.78 + 53.84 = 2207.62$.

Thus, with quarterly compounding, there will be \$2207.62 after one year.

EXERCISE 9.

1. Simple interest:

   After first year: $5000 + (5000)(0.08)(1) = 5400$

   After second year: $5400 + (5400)(0.08)(1) = 5400 + 432 = 5832.00$

   With simple interest, there will be \$5832.00 after two years.
2. Compounding semi-annually:

   After first half-year: $5000 + (5000)(0.08)(0.5) = 5200$

   After second half-year: $5200 + (5200)(0.08)(0.5) = 5200 + 208 = 5408$

   After third half-year: $5408 + (5408)(0.08)(0.5) = 5408 + 216.32 = 5624.32$

   After fourth half-year: $5624.32 + (5624.32)(0.08)(0.5) = 5624.32 + 224.97 = 5849.29$

   With semi-annual compounding, there will be \$5849.29 after two years.
3. Continuous compounding:

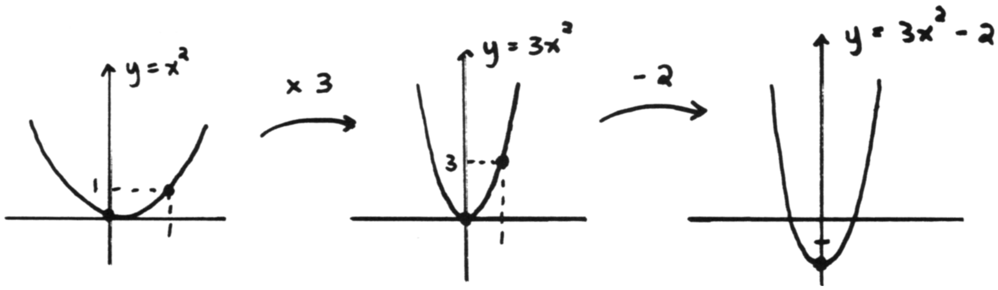
   $5000e^{(.08)(2)} = 5867.55$

   With continuous compounding, there will be \$5867.55 after two years.
4. The gain of continuous compounding over simple interest is:

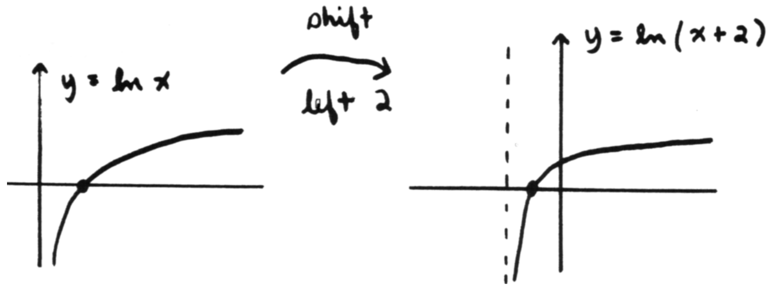
$$5867.55 - 5832.00 = 35.55$$

END-OF-SECTION EXERCISES:

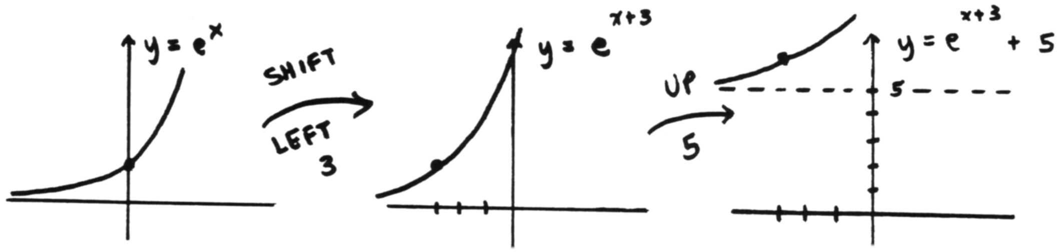
1. EXP; this is a function f^{-1}, evaluated at x
2. SEN; C. The truth of this sentence depends on the choice of function f. If f is $1-1$, then it will be true.
3. SEN; T
4. SEN; T. Any dummy variable can be used to represent a typical element of the range of f.
5. EXP
6. SEN; C. The truth depends on the choice of function f.
7. SEN; T
8. SEN; F. The sentence $e^{\ln x} = x$ is only true for nonnegative real numbers.
9. EXP
10. SEN; C. The only value of x that makes this true is $\ln 3$.
11. $\mathcal{D}(f) = \mathbb{R}$, $\mathcal{R}(f) = (-2, \infty)$



12. $\mathcal{D}(g) = (-2, \infty)$, $\mathcal{R}(g) = \mathbb{R}$



13. $\mathcal{D}(h) = \mathbb{R}$, $\mathcal{R}(h) = (5, \infty)$



14. $\mathcal{D}(f) = (4, \infty)$, $\mathcal{R}(f) = \mathbb{R}$