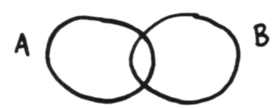
SECTION 2.3 Composite Functions

IN-SECTION EXERCISES:

EXERCISE 1.

1. a) True
   b) True
   c) False. '$A \subset B$' is a sentence that compares the sets A and B.
   d) True
   e) False. Not every rational number is an integer. For example, $\frac{1}{2} \in \mathbb{Q}$, but $\frac{1}{2} \notin \mathbb{Z}$.
   f) True. Every integer is a rational number.
   g) False. The set $\{0\}$ is NOT an element of the set $\{0,1,2\}$. Observe that the different sentence '$\{0\} \subset \{0,1,2\}$' is true.
2. An alternate way to write the set B is $B = \{3,4,5,\dots\}$.
   a) $A \cup B = \{0,1,2,3,\dots\}$
   b) $A \cap B = \{3,4,5\}$
   c) $\mathbb{R} \cap A = A$, since every element of A is a real number.
   d) $\mathbb{Z} \cap B = B$, since every element of B is an integer.
   e) $\mathbb{Q} \cap (A \cup B) = \{0,1,2,3,\dots\}$
   f) $(0,6] \cap A = \{1,2,3,4,5\}$
3. a) For the sets A and B given in 2), the sentence $A \subset B$ is false. For example, $0 \in A$, but $0 \notin B$.
   b) The sentence $B \subset A$ is also false. For example, $6 \in B$, but $6 \notin A$.
4. False. For the sets A and B given in 2), both the sentences '$A \subset B$' and '$B \subset A$' are false. The sketches below illustrate two different ways that both of these sentences can be false.



5. Since everything in C is also in $C \cup D$, the sentence '$C \subset C \cup D$' is always true.

EXERCISE 2.

1. $(f+g)(x) = f(x) + g(x) = x^2 + \sqrt{x}$
   $\mathcal{D}(f+g) = [0,\infty)$
2. $(f-g)(x) := f(x) - g(x)$
   $\mathcal{D}(f-g) = \{x \mid x \in \mathcal{D}(f) \text{ and } x \in \mathcal{D}(g)\} = \mathcal{D}(f) \cap \mathcal{D}(g)$
3. $(fg)(x) := f(x) \cdot g(x)$
   $\mathcal{D}(fg) = \{x \mid x \in \mathcal{D}(f) \text{ and } x \in \mathcal{D}(g)\} = \mathcal{D}(f) \cap \mathcal{D}(g)$

EXERCISE 3.

1. The dot '$\cdot$' denotes multiplication of real numbers.
2. The function kf takes an input x, applies f to it to get $f(x)$, and then multiplies this number $f(x)$ by k.
3. $\mathcal{D}(kf) = \mathcal{D}(f)$
4. For $f(x) = x^3$ and $k = 4$, $(kf)(x) = 4x^3$.

EXERCISE 4.

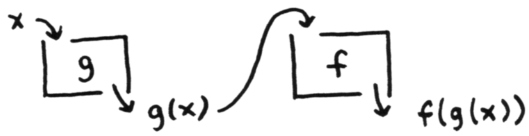
1. There are two things to worry about: f must know how to act on x, and $f(x)$ must be nonnegative. Thus:

$$\mathcal{D}(\sqrt{f}) = \{x \mid x \in \mathcal{D}(f) \text{ and } f(x) \geq 0\}$$

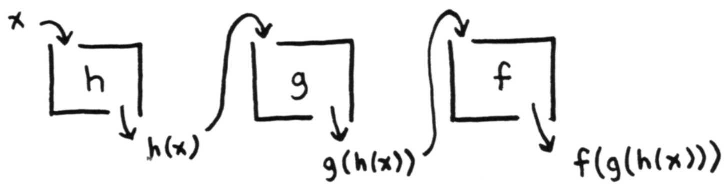
2. If $f(x) = x^3$, then $(\sqrt{f})(x) = \sqrt{x^3}$. In order for x^3 to be nonnegative, x must be nonnegative. Thus, $\mathcal{D}(\sqrt{f}) = [0, \infty)$.
3. If $g(x) = -x^2$, then $(\sqrt{g})(x) = \sqrt{-x^2}$. In order for $-x^2$ to be nonnegative, x must be zero. Thus, $\mathcal{D}(\sqrt{g}) = \{0\}$. (Pretty uninteresting!)

EXERCISE 5.

1. By definition, $(f \circ g)(x) = f(g(x))$. The boxes below describe the action of this function:



2. In the function $f \circ g \circ h$, h acts first (on x), g acts second (on $h(x)$), and f acts third (on $g(h(x))$). The boxes below describe this function:



EXERCISE 6.

1. The function f takes an input, adds 1, then squares the result.
2. The series of boxes below describe f:

3. Define $A(x) := x + 1$ and $S(x) = x^2$. Then:

$$\begin{aligned}(S \circ A)(x) &= S(A(x)) \\ &= S(x+1) \\ &= (x+1)^2 \\ &= f(x)\end{aligned}$$

EXERCISE 7.

1.
$$\begin{aligned}\mathcal{D}(g \circ f) &= \{x \mid x \in \mathcal{D}(f) \text{ and } f(x) \in \mathcal{D}(g)\} \\ &= \{x \mid x \in \mathbb{R} \text{ and } -x^3 \in [0, \infty)\} \\ &= (-\infty, 0]\end{aligned}$$

EXERCISE 8.

1. $(f \circ g)(x) = f(g(x)) = f(\frac{1}{x}) = |\frac{1}{x}| = \frac{1}{|x|}$
   $\mathcal{D}(f \circ g) = \{x \mid x \neq 0\}$
2. $(g \circ f)(x) = g(f(x)) = g(|x|) = \frac{1}{|x|}$
   $\mathcal{D}(g \circ f) = \{x \mid x \neq 0\}$

3. In this case, the functions $f \circ g$ and $g \circ f$ are the same.
4. The functions $f \circ g$ and $g \circ f$ need not always be the same. In the previous text example, where $f(x) = -x^3$ and $g(x) = \sqrt{x}$, $f \circ g \neq g \circ f$.

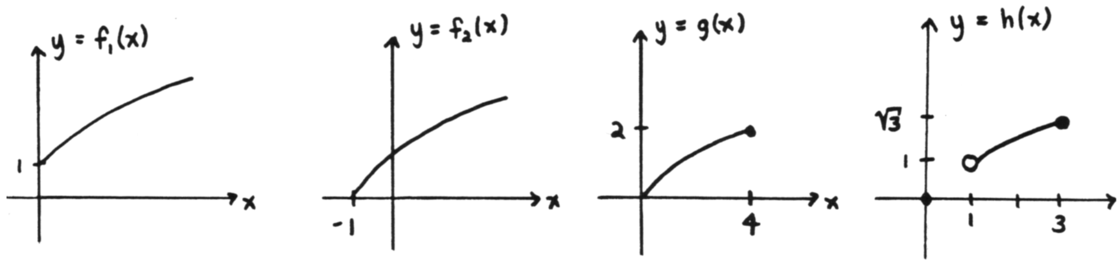
EXERCISE 9.

Some quick sketches might be helpful in getting the range information for the following functions. These functions are graphed below.

1. $\mathcal{D}(f_1) = [0, \infty)$, $\mathcal{R}(f_1) = [1, \infty)$
2. $\mathcal{D}(f_2) = [-1, \infty)$, $\mathcal{R}(f_2) = [0, \infty)$
3. $\mathcal{D}(g) = [0, 4]$; just read this information off!

$$\begin{aligned} \mathcal{R}(g) &= \{g(x) \mid x \in \mathcal{D}(g)\} \\ &= \{\sqrt{x} \mid x \in [0,4]\} \\ &= [0, 2] \end{aligned}$$

4. $\mathcal{D}(h) = \{0\} \cup (1, 3]$
   $\mathcal{R}(h) = \{0\} \cup (1, \sqrt{3}]$



END-OF-SECTION EXERCISES:

1. EXP; $A \cup B$ is a set
2. SEN; TRUE. The sentence '$A \subset A \cup B$' is true for all sets A and B.
3. SEN; CONDITIONAL. The truth of the sentence '$A \subset B$' depends upon the sets A and B.
4. EXP; $\mathcal{R}(f)$ is a set; it is the range of some function f.
5. SEN; CONDITIONAL. The sentence '$\mathcal{R}(f) = \mathbb{R}$' states that the range of a function is the set of real numbers; the truth of this sentence depends upon the function f being referred to.
6. EXP; this is a set
7. SEN; CONDITIONAL. The truth of this sentence depends upon the choice of functions f and g, and the choice of x.
8. SEN; TRUE, by definition. Remember that the symbol ':=' means 'equals, by definition'. Here, the function $f + g$ is being defined.
9. SEN; FALSE. The set $\{a\}$ is NOT an element of the set $\{a, b\}$.
10. SEN; TRUE, since $a \in \{a, b\}$
11. $\mathcal{R}(f) = [0, 8]$
12. $\mathcal{R}(g) = [-1, \infty)$
13. $\mathcal{R}(h) = \{1, 2, 3, 4, 5\}$
14. $\mathcal{R}(f) = \{2, 3, 4\}$