SECTION 2.1 Functions and Function Notation

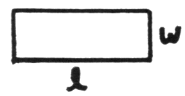
IN-SECTION EXERCISES:

EXERCISE 1.

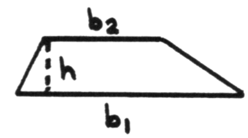
1. The input is naturally viewed as 'r', the radius of the circle. The output is naturally viewed as 'A'. Rewriting, for all $r \geq 0$:

$$\begin{aligned} A = \pi r^2 \quad &\iff \quad r^2 = \frac{A}{\pi} \\ &\iff \quad r = \sqrt{\frac{A}{\pi}} \end{aligned}$$

Now, the radius is the 'natural' output.



2. Area of a rectangle: $A = lw$ (Many others are possible.)



3. Area of a trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$ (Many others are possible.)

EXERCISE 2.

1. input 5; output $5^2 = 25$
   input -5; output $(-5)^2 = 25$
   input x; output x^2
   input x^2; output $(x^2)^2 = x^4$
   input $x + h$; output $(x + h)^2 = x^2 + 2xh + h^2$
2. inputs 3 and 4; output $3 + 4 = 7$
   inputs x^2 and y^2; output $x^2 + y^2$
   inputs t and t; output $t + t = 2t$

EXERCISE 3.

First graph: y is a function of x; x is not a function of y

Second graph: y is not a function of x; x is a function of y

Third graph: y is a function of x; x is a function of y

EXERCISE 4.

1.

$$\begin{aligned} x^2 + y^2 = 9 \quad &\iff \quad x^2 = 9 - y^2 \\ &\iff \quad x = \pm\sqrt{9 - y^2} \end{aligned}$$

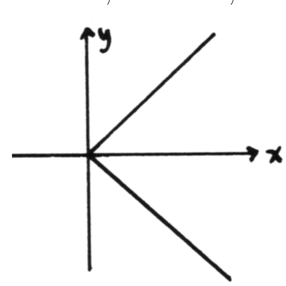
2. Allowable values for y:

$$\begin{aligned} 9 - y^2 \geq 0 \quad &\iff \quad y^2 \leq 9 \\ &\iff \quad |y| \leq 3 \\ &\iff \quad -3 \leq y \leq 3 \end{aligned}$$

3. For each allowable value of y, there are TWO values of x; therefore, x is not a function of y.

EXERCISE 5.

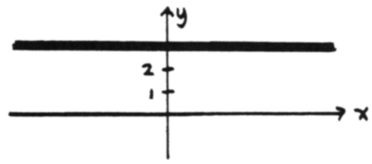
1. The graph of $x = |y|$ is shown at right:



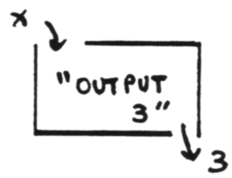
2. Here, y is not a function of x. There are many x-values that have 2 associated y-values.
3. Here, x is a function of y. Every y-value has a unique associated x-value.

EXERCISE 6.

1. The graph of $y = 3$, viewed as an equation in 2 variables, is shown below.



2. Here, y is a function of x. For every x, there is a unique value of y.
3. However, x is NOT a function of y. For the one allowable value of y, there are an *infinite* number of associated x-values. This is about as far from a function as you can get!
4. The corresponding 'black box' is one that always outputs a 3, no matter what the input is:



EXERCISE 7.

1. First way: $f(x) = 2x + 3$

   Second way: $g(x) = 2x + 3$

   Third way: $g(t) = 2t + 3$

   When $x + y$ is the input, $2(x + y) + 3$ is the output.
2. First way: $f(x, y) = \frac{1}{2}(x + y)$

   Second way: $h(a, b) = \frac{1}{2}(a + b)$

   Third way: $A(t, w) = \frac{1}{2}(t + w)$

   When the inputs are x and $5x$, the output is: $\frac{1}{2}(x + 5x) = \frac{1}{2}(6x) = 3x$

EXERCISE 8.

First box: $f(x) = \sqrt{x} + 2$

Second box: $M(a, b) = ab$

Third box: $D(x, y, z) = 2x + 2y + 2x$

EXERCISE 9.

1. $d(0) = \frac{f(0+h)-f(0)}{h} = \frac{f(h)-f(0)}{h}$; this cannot be written any more simply, unless the function values $f(h)$ and $f(0)$ are known.
2. $d(y) = \frac{f(y+h)-f(y)}{h}$
3. $d(x + h) = \frac{f((x+h)+h)-f(x+h)}{h} = \frac{f(x+2h)-f(x+h)}{h}$
4. $d(x^2) = \frac{f(x^2+h)-f(x^2)}{h}$
5. $D(0, 1) = \frac{f(0+1)-f(0)}{1} = f(1) - f(0)$
6. $D(y, k) = \frac{f(y+k)-f(y)}{k}$
7. $D(x + \epsilon, k) = \frac{f(x+\epsilon+k)-f(x+\epsilon)}{k}$
8. $D(x + h, 2h) = \frac{f((x+h)+h)-f(x+h)}{2h} = \frac{f(x+2h)-f(x+h)}{2h}$

EXERCISE 10.

1. When x is 3, the sentence '$x = 3$ or $x = -3$' is true, since the component sentence '$x = 3$' is true.
2. The solution set of '$x = 3$ or $x = -3$' is $\{3, -3\}$.
3. When x is 4, the sentence '$x = 3$ or $x = -3$' becomes '$4 = 3$ or $4 = -3$', which is false.

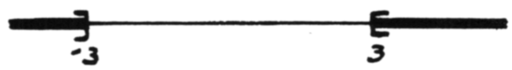
EXERCISE 11.

1. $(-1, 0) \cup [0, 2) = (-1, 2)$
2. $\{x \mid x \text{ is rational}\} \cup \{x \mid x \text{ is irrational}\} = \mathbb{R}$
3. $\{-1, 2, 100\} \cup \mathbb{Z} = \mathbb{Z}$
4. $\{t \mid t \geq 0\} \cup \{x \mid x < 0\} = \mathbb{R}$
5. The set shown is: $(-1, 3] \cup [4, \infty)$
6. The set shown is: $\{-1\} \cup (3, 5)$
7. The set shown is: $(-\infty, 3] \cup \{5, 7\}$

EXERCISE 12.

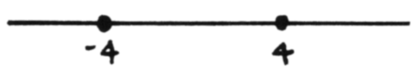
1. $|x| < 2 \iff -2 < x < 2$

2. $|t| \geq 3 \iff t \geq 3 \text{ or } t \leq -3$



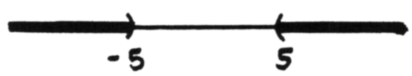
3.

$$\begin{aligned} 5 - |x| = 1 &\iff -|x| = -4 \\ &\iff |x| = 4 \\ &\iff x = \pm 4 \end{aligned}$$



4.

$$\begin{aligned} 2 - |t| < -3 &\iff -|t| < -5 \\ &\iff |t| > 5 \\ &\iff t > 5 \text{ or } t < -5 \end{aligned}$$



5. The numbers shown are precisely those whose distance from 0 is greater than 1; this is the solution set of $|x| > 1$.
6. The numbers shown are precisely those whose distance from 0 is less than or equal to 1; this is the solution set of $|x| \leq 1$.
7. The numbers shown are precisely those whose distance from 0 is 2; this is the solution set of $|x| = 2$.

EXERCISE 13.

1.

$$\begin{aligned} t^2 = 7 &\iff |t| = \sqrt{7} \\ &\iff t = \pm\sqrt{7} \end{aligned}$$

2.

$$
\begin{aligned}
(2t-5)^2 = 3 \quad &\iff \quad |2t-5| = \sqrt{3} \\
&\iff \quad 2t-5 = \pm\sqrt{3} \\
&\iff \quad 2t = 5 \pm \sqrt{3} \\
&\iff \quad t = \frac{1}{2}(5 \pm \sqrt{3})
\end{aligned}
$$

3.

$$
\begin{aligned}
x^2 < 4 \quad &\iff \quad |x| < 2 \\
&\iff \quad -2 < x < 2
\end{aligned}
$$

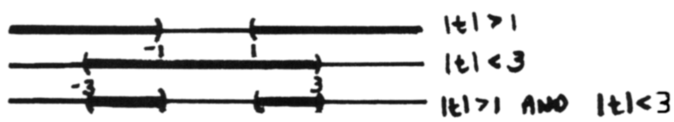
4.

$$
\begin{aligned}
x^2 + 6x + 9 > 4 \quad &\iff \quad (x+3)^2 > 4 \\
&\iff \quad |x+3| > 2 \\
&\iff \quad x+3 > 2 \ \text{ or } \ x+3 < -2 \\
&\iff \quad x > -1 \ \text{ or } \ x < -5
\end{aligned}
$$

5.

$$
\begin{aligned}
(|t|-2)^2 < 1 \quad &\iff \quad \big|\, |t| - 2 \,\big| < 1 \\
&\iff \quad -1 < |t| - 2 < 1 \\
&\iff \quad -1 < |t| - 2 \ \text{ and } \ |t| - 2 < 1 \\
&\iff \quad 1 < |t| \ \text{ and } \ |t| < 3 \\
&\iff \quad t \in (1,3) \cup (-3,-1)
\end{aligned}
$$

The last step is explained by the sketch below:



EXERCISE 14.

1. THEOREM. For all real numbers z, and for $a \ge 0$:

$$
z^2 \ge a \quad \iff \quad |z| \ge \sqrt{a} \quad \iff \quad z \ge \sqrt{a} \ \text{ or } \ z \le -\sqrt{a}
$$

Using this theorem:

$$
\begin{aligned}
(2x-1)^2 \ge 3 \quad &\iff \quad |2x-1| \ge \sqrt{3} \\
&\iff \quad 2x-1 \ge \sqrt{3} \ \text{ or } \ 2x-1 \le -\sqrt{3} \\
&\iff \quad 2x \ge 1+\sqrt{3} \ \text{ or } \ 2x \le 1-\sqrt{3} \\
&\iff \quad x \ge \frac{1}{2}(1+\sqrt{3}) \ \text{ or } \ x \le \frac{1}{2}(1-\sqrt{3})
\end{aligned}
$$

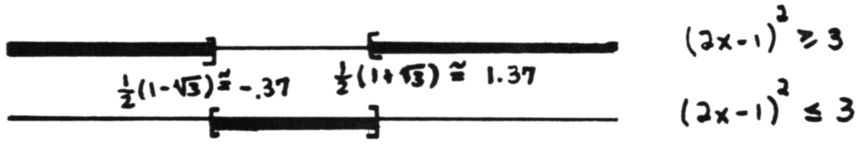
2. THEOREM. For all real numbers z, and for $a \ge 0$:

$$z^2 \le a \iff |z| \le \sqrt{a} \iff -\sqrt{a} \le z \le \sqrt{a}$$

Using this theorem:

$$\begin{aligned}(2x-1)^2 \le 3 &\iff |2x-1| \le \sqrt{3} \\ &\iff -\sqrt{3} \le 2x-1 \le \sqrt{3} \\ &\iff 1-\sqrt{3} \le 2x \le 1+\sqrt{3} \\ &\iff \frac{1}{2}(1-\sqrt{3}) \le x \le \frac{1}{2}(1+\sqrt{3})\end{aligned}$$

The solution sets of both $(2x-1)^2 \ge 3$ and $(2x-1)^2 \le 3$ are shown below. Since *every* real number is either ≥ 3 or ≤ 3, the solution sets union to $\mathbb{R}$. Also, the only real number that is BOTH ≥ 3 and ≤ 3 is 3 itself; and the only time that $(2x-1)^2$ equals 3 is when x is $\frac{1}{2}(1 \pm \sqrt{3})$. These are the only two numbers where the solution sets overlap.



EXERCISE 15.

1.

$$\begin{aligned}y^2 - x^2 = 1 &\iff y^2 = x^2 + 1 \\ &\iff y = \pm\sqrt{x^2+1}\end{aligned}$$

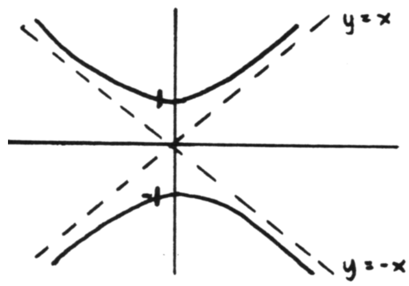
2.

$$\begin{aligned}y^2 - x^2 = 1 &\iff -x^2 = 1 - y^2 \\ &\iff x^2 = y^2 - 1 \\ &\iff x = \pm\sqrt{y^2-1}\end{aligned}$$

3. If (x, y) is a pair of real numbers that makes the sentence '$y^2 - x^2 = 1$' true, then we must have $y^2 - 1 \geq 0$. That is:

$$\begin{aligned} y^2 - 1 \geq 0 &\iff y^2 \geq 1 \\ &\iff |y| \geq 1 \\ &\iff y \geq 1 \text{ or } y \leq -1 \end{aligned}$$

Indeed, you might recall from an analytic geometry course that the graph of $y^2 - x^2 = 1$ is the hyperbola shown below:



END-OF-SECTION EXERCISES:

1. $f(0) = 0^3 - 1 = -1$
   $f(1) = 1^3 - 1 = 0$
   $f(-1) = (-1)^3 - 1 = -1 - 1 = -2$
   $f(t) = t^3 - 1$
   $f(f(2))$; first find $f(2)$: $f(2) = 2^3 - 1 = 7$; then, $f(f(2)) = f(7) = 7^3 - 1 = 342$

2. $g(x + h) = -(x + h)^4 + (x + h)$
   $g(-x) = -(-x)^4 + (-x) = -x^4 - x$
   $g(-1) = -(-1)^4 + (-1) = -1 - 1 = -2$

3. $f(-2) = |-2| = 2$

   $$f(t) = |t| = \begin{cases} t & \text{if } t \geq 0 \\ -t & \text{if } t < 0 \end{cases}$$

   $f(-t) = |-t| = |t|$
   $f(x^2) = |x^2| = |x|^2$

4. $g(-x) = |-x - 2| = |(-1)(x + 2)| = |x + 2|$
   $g(|t|) = \big|\, |t| - 2 \,\big|$
   $g(\sqrt{2}) = |\sqrt{2} - 2|$
   $g(x + 2) = |(x + 2) - 2| = |x|$

5. $h(-x) = \frac{1}{-x} = -\frac{1}{x}$
   $h(h(x)) = h(\frac{1}{x}) = \frac{1}{1/x} = x$
   $h(\frac{1}{x}) = \frac{1}{1/x} = x$
   $h(x + \Delta x) = \frac{1}{x+\Delta x}$
   $h(|x|) = \frac{1}{|x|} = \left|\frac{1}{x}\right|$

6. $h(t) = \sqrt{t^2 - 1}$

   $h(x + \Delta x) = \sqrt{(x + \Delta x)^2 - 1}$

   $h(-x) = \sqrt{(-x)^2 - 1} = \sqrt{x^2 - 1}$

   $h(1) = \sqrt{1^2 - 1} = \sqrt{0} = 0$

7. $h(1, 1) = 1^2 + 1^2 - 1 = 1$

   $h(x, x) = x^2 + x^2 - 1 = 2x^2 - 1$

   $h(y, x) = y^2 + x^2 - 1 = h(x, y)$

   $h(x + \Delta x, y + \Delta y) = (x + \Delta x)^2 + (y + \Delta y)^2 - 1$

8. $h(0, 0)$ is not defined; it produces division by zero

   $h(y, x) = \dfrac{1}{y(x - 1)}$

   $h(x^2, y) = \dfrac{1}{x^2(y - 1)}$

   $h(x, y^2) = \dfrac{1}{x(y^2 - 1)}$