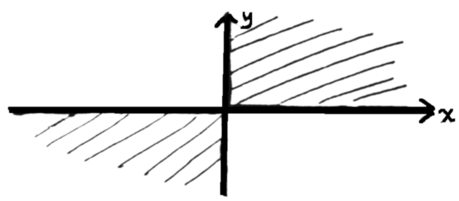
SECTION 1.4 Mathematical Equivalence

IN-SECTION EXERCISES:

EXERCISE 1.

1. The implied domain of '$\frac{x}{y} = 1$' is $\{(x, y) \,|\, y \neq 0\}$.
2. By convention, the only variable in the sentence '$ax + b = 0$' is x; a and b are constants. The implied domain is $\mathbb{R}$.
3. The sentence '$\sqrt{x^2} - 4 = 0$' is defined for *all* real numbers x, since x^2 is always greater than or equal to zero. The implied domain is $\mathbb{R}$.
4. The sentence '$\sqrt{x^3} = 2$' is only defined when $x^3 \geq 0$. The implied domain is $\{x \,|\, x \geq 0\}$.
5. The sentence '$\sqrt[4]{xy} - x = y - 5$' is defined only when $xy \geq 0$. This happens only when x and y are both nonnegative, or both nonpositive. The domain is shaded below.



EXERCISE 2.

1. Each sentence has implied domain $\mathbb{R}$. Both are true when $x = 2$ and false otherwise. They are equivalent.
2. Each sentence has implied domain $\mathbb{R}$. The sentence '$x^2 - 16 = 0$' has solution set $\{4, -4\}$; the sentence '$x = 4$' has solution set $\{4\}$. Thus, when x is -4, the first sentence is true, but the second is false. They are not equivalent.
3. Each sentence has implied domain $\mathbb{R}$. The sentence '$|x| = 3$' has solution set $\{3, -3\}$; the sentence '$x = 3$' has solution set $\{3\}$. Thus, when x is -3, the first sentence is true, but the second is false. They are not equivalent.
4. Each sentence has implied domain $\mathbb{R}$. The sentence '$|x| = 3$' has solution set $\{3, -3\}$; the sentence '$x = 3$ and $x = -3$' has empty solution set—there is no real number which is simultaneously equal to 3, and not equal to 3. In particular, when x is 3, the first sentence is true, but the second is false. They are not equivalent.
5. Each sentence has implied domain $\mathbb{R}$. Both sentences have solution set $(0, \infty)$. They are equivalent.
6. Each sentence has implied domain $\mathbb{R}$. Both sentences have solution set $(0, \infty)$. They are equivalent.
7. Each sentence has implied domain $\mathbb{R}$. The sentence '$x^2 \geq 0$' has solution set $\mathbb{R}$; the sentence '$x \geq 0$' has solution set $[0, \infty)$. In particular, when x is -1, the first sentence is true, but the second is false. They are not equivalent.
8. Each sentence has implied domain $\mathbb{R}$. Both sentences have solution set $(1, 3]$. They are equivalent.

EXERCISE 3.

They can all be solved by inspection; but some require more careful inspection than others!

1. For '$7x - 3 = 0$' to be true, $7x$ must equal 3; thus, x must equal $\frac{3}{7}$.
2. The sentence '$x^2 \geq 0$' is true for all real numbers.
3. Any real number, when squared, is greater than or equal to zero; thus the solution set of $(x + 1)^2 \geq 0$ is $\mathbb{R}$.
4. The sentence '$x = -0.2$' is very easily solved by inspection; the only number that equals -0.2 is -0.2.
5. The sentence '$x < 0$' is very easily solved by inspection; the only numbers less than zero are $x \in (-\infty, 0)$.
6. The sentence '$x^3 < 0$' is true for all real numbers with cubes that are negative; thus it is true for all negative numbers.

EXERCISE 4.

1. A *theorem* is a mathematical result that is both *true* and *important*.

2. This theorem tells us that adding the same number to both sides of an equation yields an equivalent equation. More precisely, the sentence '$x = y$' will always have the same truth value (T or F) as '$x + z = y + z$'. Here, the letters x, y and z have been used to denote typical real numbers, instead of a, b, and c.

   When $x = 3$, $y = 4$ and $z = 5$, the sentence '$x = y$' becomes '$3 = 4$', which is false. The sentence '$x + z = y + z$' becomes '$3 + 5 = 4 + 5$', which is also false.
3. This 'theorem' doesn't make any sense. The reader is told that a, b and c are real numbers, but then nothing further is said about them! And, the reader has not been given any information about what x, y and z are!

EXERCISE 5.

1. If you add the same number to both sides of an inequality, you get an equivalent inequality in the same direction. More precisely, for any choices of a, b, and c, the sentences '$a < b$' and '$a + c < b + c$' will always have the same truth values.
2. When $a = 1$, $b = 2$ and $c = 3$, the sentence '$a < b$' becomes '$1 < 2$', which is true. The sentence '$a + c < b + c$' becomes '$1 + 3 < 2 + 3$', which is also true.
3. When $a = 2$, $b = 1$ and $c = 3$, the sentence '$a < b$' becomes '$2 < 1$', which is false. The sentence '$a + c < b + c$' becomes '$2 + 3 < 1 + 3$', which is also false.
4. See (1).

EXERCISE 6.

1. If you multiply both sides of an inequality by a negative number, you must change the sense of the inequality (in order to preserve the truth value).
2. See (1).
3. THEOREM. For all real numbers a and b, and for $c > 0$:

$$a < b \quad \iff \quad ac < bc$$

EXERCISE 7.

$$\begin{aligned} 3x - 7 = 1 \quad &\iff \quad 3x = 8 \qquad \text{(add 7)} \\ &\iff \quad x = \frac{8}{3} \qquad \text{(divide by 3)} \end{aligned}$$

EXERCISE 8.

$$\begin{aligned} 3x - 7 < 1 \quad &\iff \quad 3x < 8 \qquad \text{(add 7)} \\ &\iff \quad x < \frac{8}{3} \qquad \text{(divide by 3)} \end{aligned}$$

EXERCISE 9.

$$\begin{aligned} 3(x + 2) = (3x + 1) + 4 \quad &\iff \quad 3x + 6 = 3x + 5 \qquad \text{(simplify)} \\ &\iff \quad 6 = 5 \qquad \text{(subtract } 3x\text{)} \end{aligned}$$

Since the equation $6 = 5$ is never true, neither is the original equation.

EXERCISE 10.

$$\begin{aligned} 2x-(7-x)=x+1-2(4-x) &\iff 2x-7+x=x+1-8+2x && \text{(simplify)} \\ &\iff 3x-7=3x-7 && \text{(simplify)} \\ &\iff 0=0 && \text{(subtract } 3x-7\text{)} \end{aligned}$$

Since the equation $0 = 0$ is always true, so is the original equation.

END-OF-SECTION EXERCISES:

2. SEN, C; true if x is 4, and false otherwise
3. SEN, T. Both sentences have the same implied domain, and the same solution set: $\{4\}$
4. SEN, T. Both sentences have the same implied domain (all nonzero real numbers) and the same solution set: $\{\frac{1}{3}\}$
5. SEN, T. Both sentences have the same implied domain, and the same solution set: $\{0\}$
6. SEN, F. Both sentences have the same implied domain. However, the first has solution set $\{1, -1\}$; the second has solution set $\{1\}$. Thus, when y is -1, the first sentence is true, but the second is false.
7. EXP
8. SEN, T. Both sentences have the same implied domain, and the same solution set: $\{2\}$
9. SEN, T. Both sentences have the same implied domain, and the same solution set: $\{-2\}$
10. SEN, F. Both sentences have the same implied domain. However, the first has solution set $\{0, 1\}$; the second has solution set $\{1\}$. Thus, when x is 0, the first sentence is true, but the second is false.
11.

$$\begin{aligned} 5x-7=3 &\iff 5x=10 && \text{(add 7)} \\ &\iff x=2 && \text{(divide by 5)} \end{aligned}$$

12.

$$\begin{aligned} 5-3y=9 &\iff -3y=4 && \text{(subtract 5)} \\ &\iff y=-\frac{4}{3} && \text{(divide by } -3\text{)} \end{aligned}$$

13.

$$\begin{aligned} 3x<x-11 &\iff 2x<-11 && \text{(subtract } x\text{)} \\ &\iff x<-\frac{11}{2} && \text{(divide by the positive number 2)} \end{aligned}$$

14.

$$\begin{aligned} 3t+7\ge -2 &\iff 3t\ge -9 && \text{(subtract 7)} \\ &\iff t\ge -3 && \text{(divide by the positive number 3)} \end{aligned}$$