SECTION 1.2 The Role of Variables

IN-SECTION EXERCISES:

EXERCISE 1.

1. TRUE
2. TRUE
3. CONDITIONAL; true when x is 3, false otherwise
4. FALSE; the correct statement is $\sqrt{9} = 3$
5. CONDITIONAL; true for all ordered pairs (x, y) of the form $(t, 4-t)$, where t is any real number; false otherwise
6. FALSE; the correct statement is $\sqrt{(-9)^2} = |-9| = 9$
7. CONDITIONAL; true for all nonzero real numbers x, false when x is zero

EXERCISE 2.

1. What number has the property that when 8 is subtracted from it, the result is 6? ANS: 14. Thus, twice the desired number must equal 14. The desired number must be 7.
2. What number(s), when squared, equal 9? Both 3 and -3. In order for $x - 2$ to equal 3, x must equal 5. In order for $x - 2$ to equal -3, x must equal -1. The solutions are 5 and -1.

EXERCISE 4.

1. With universal set $\mathbb{R}$, the solutions are 1 and -1.
2. With universal set $\mathbb{C}$, the solutions are 1, -1, i, and $-i$.
3. With universal set the integers, the solutions are 1 and -1.

EXERCISE 5.

1. $\sqrt{2}$, $-\sqrt{2}$
2. There are no solutions in the rational numbers.
3. $\sqrt{2}$, $-\sqrt{2}$
4. There are no solutions in the integers.

EXERCISE 6.

There are various statements of the Fundamental Theorem of Algebra, but they all amount to the same thing: *every* polynomial equation of degree n must have n solutions in $\mathbb{C}$ (counting multiplicities).

The phrase 'counting multiplicities' means that a solution may be repeated: for example, $x^2 - 4x + 4 = (x-2)(x-2) = 0$ has only the solution 2, but it appears twice (once for each factor).

Thus, the polynomial equation $x^6 = 1$ (degree 6) must have 6 solutions in $\mathbb{C}$, even though there are only two solutions in $\mathbb{R}$. (It ends up that these six solutions are equally–spaced on the unit circle in the complex plane.)

EXERCISE 7.

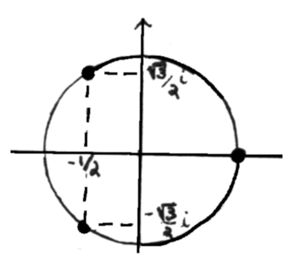
1. $\sqrt{3}w - \frac{1}{2}z = 5$; there are many possible examples here
2. $x^2 - 2x + \sqrt{x} > 0$; there are many possible examples here
3. the only variable is x; α, β and γ are constants

EXERCISE 8.

1. The four variables are: x_1, x_2, x_3 and x_4
2. A solution is a 4-tuple (a, b, c, d) such that when a is substituted for x_1, b for x_2, etc., a true statement results.
3. $(0, 0, 0, 0)$ is a solution since $0 + 2(0) + 3(0) + 0 = 0$
4. To find another solution, choose *any* numbers you want for any three of the variables; then, determine what the fourth number must be to make the equation true. For example, choose $x_1 = x_2 = 0$ and $x_3 = 1$; then x_4 must equal -3. Thus, $(0, 0, 1, -3)$ is a solution. There are an infinite number of solutions.

END-OF-SECTION EXERCISES:

1. EXP
2. SEN, T
3. SEN, F
4. SEN, T
5. SEN, T
6. SEN, T
7. SEN, F
8. SEN, T
9. SEN, F. There is only one variable, x.
10. SEN, F. The only variable is x. By convention, a, b and c are constants.
11. SEN, T
12. SEN, F. In an n-tuple, the order is important.
13. EXP
14. SEN, C. For example, it is true when both y and z are 0. It is false if $y = 2$ and $z = 1$.
15. $\mathbb{R}$: the only solution is 1
    the rational numbers: the only solution is 1
    the integers: the only solution is 1
16. $\mathbb{R}$: the solutions are $\pm\sqrt{7}$
    the rational numbers: there are no rational solutions
    the integers: there are no integer solutions
17. $\mathbb{R}$: setting each factor to 0, the real number solutions are 1, $-\pi$, and $\frac{3}{2}$
    the rational numbers: the only rational solutions are 1 and $\frac{3}{2}$
    the integers: the only integer solution is 1
18. $\mathbb{R}$: the real number solutions are 0, ± 2, and $\pm\sqrt{2}$
    the rational numbers: the rational solutions are 0 and ± 2
    the integers: the integer solutions are 0 and ± 2
19. a) The points are plotted below:



    b) Since $(-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2 = \frac{1}{4} + \frac{3}{4} = 1$, the point $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ lies on the unit circle. Same for the remaining point.

c) Clearly, the number 1 satisfies the equation. To see that $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ satisfies $x^3 = 1$, observe that:

$$\begin{aligned}
(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)^3 &= (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) \\
&= (\frac{1}{4} - \frac{\sqrt{3}}{2}i - \frac{3}{4})(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) \\
&= (-\frac{1}{2} - \frac{\sqrt{3}}{2}i)(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) \\
&= \frac{1}{4} + \frac{3}{4} \\
&= 1
\end{aligned}$$

Similarly for the remaining number.

d) The equation $x^3 - 1 = 0$ has the same solutions as the equation $x^3 = 1$, so the problem has already been solved.