SOLUTIONS TO ALL IN-SECTION
and END-OF-SECTION EXERCISES

CHAPTER 1. ESSENTIAL PRELIMINARIES

SECTION 1.1 The Language of Mathematics—Expressions versus Sentences

IN-SECTION EXERCISES:

EXERCISE 1. A mathematical *expression* is just a name given to some mathematical object of interest; it is the analogue of a English noun. 'Objects of interest' in mathematics are frequently numbers, functions, or sets. Some examples of expressions: x, $1+\sqrt{3}$, $\frac{t}{x}+\pi$

A mathematical *sentence* must express a complete thought; sentences have verbs. For example, $x=1$ is a sentence (the verb is '$=$'). Sentences can be TRUE, like $1+1=2$ and $x^2\ge 0$. Note that $x^2\ge 0$ is true for *all* real numbers x. Sentences can be FALSE, like $1+1=3$ and $x^2<0$. Note that $x^2<0$ is false for *all* real numbers x. Sentences can be SOMETIMES TRUE/SOMETIMES FALSE, like $x=1$ and $|t|=t$. Note that $x=1$ is true only when x is 1, and is false otherwise; $|t|=t$ is true only for $t\ge 0$, and false otherwise.

EXERCISE 2.

1. Most algebra students are taught to solve $2x-1=0$ by writing down a list of equations such as these:

$$\begin{aligned} 2x-1&=0 &&\text{(start with original equation)}\\ 2x&=1 &&\text{(add 1 to both sides)}\\ x&=\frac{1}{2} &&\text{(divide both sides by 2)}\end{aligned}$$

   The student concludes that the only number that makes $2x-1=0$ true is $\frac{1}{2}$.

   What is the connection between the three equations above? This idea is discussed in the next few sections.

2. To solve the inequality $-3t<2$, most students write:

$$\begin{aligned} -3t&<2 &&\text{(start with original inequality)}\\ t&>-\frac{2}{3} &&\text{(divide both sides by -3; change sense of inequality)}\end{aligned}$$

   The numbers that make $-3t<2$ true are all numbers greater than $-\frac{2}{3}$.

3. You get:

$$\begin{aligned} y+y+y&=3y &&\text{(start with original equation)}\\ 0&=0 &&\text{(subtract $3y$ from both sides)}\end{aligned}$$

   Note that the resulting equation $0=0$ is *always true.*

4. Since the correct expansion is

$$(a+b)^2=(a+b)(a+b)=a^2+2ab+b^2\ ,$$

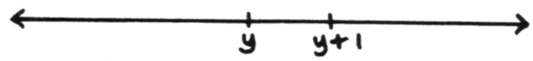
   the only time that $(a+b)^2=a^2+b^2$ is true is when $2ab=0$. This happens when $a=0$ or when $b=0$.

   Once we develop more language skills, we will be able to write this solution more compactly.

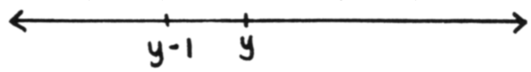
5. An English sentence that is *intentionally* false is called a LIE. One that is *nonintentionally* false is probably called a MISTAKE!

EXERCISE 3.

1. '$a + b = b + a$'; SEN, T. This is the commutative property of addition.
2. '$a + b$'; EXP
3. '$a + b = 5$'; SEN, ST/SF. We will see that there is an entire LINE of points (a, b) that make this true.
4. 'rectangle'; NOUN
5. 'Every rectangle has three sides.'; SEN, F. Rectangles have four sides.
6. '$x + (-x) \neq 0$'; SEN, F. For all real numbers x, $x + (-x) = 0$.
7. '$3 \leq 3$'; SEN, T
8. '$y \geq y$'; SEN, T
9. '$y > y + 1$'; SEN, F. The number y always lies to the *left* of $y + 1$ on the number line.



10. '$y > y - 1$'; SEN, T. The number y always lies to the right of $y - 1$ on the number line.



11. 'Bob'; NOUN
12. 'Bob has red hair.'; SEN, ST/SF. If the 'Bob' being referred to does indeed have red hair, then this is true. Otherwise, it is false.
13. 'For all nonzero real numbers x, $x^0 = 1$.'; SEN, T. This is the definition of a zero exponent.
14. 'The distance between real numbers a and b is $b - a$.'; SEN, ST/SF. If $b \geq a$, then the distance between them is $b - a$. But if $b < a$, then the distance is $a - b$. A correct formula for the distance, that works for all choices of a and b, is $|b - a|$.
15. '$a(b + c)$'; EXP
16. '$a(b + c) = ab + ac$'; SEN, T. This is the Distributive Law (of multiplication over addition).

EXERCISE 4.

1. TRUE: Julia begins with the letter 'J'.

   FALSE: Julia begins with the letter 'G'.

   SOMETIMES TRUE/SOMETIMES FALSE: Julia has red hair.
2. TRUE: $x^2 + y^2 \geq 0$

   FALSE: $x^2 + y^2 < 0$

   SOMETIMES TRUE/SOMETIMES FALSE: $x^2 + y^2 > 0$

EXERCISE 6.

1. $5 = \frac{5}{1} = 1 + 4 = 1.1 + 3.9 = \frac{10}{2} = \sqrt{25} = 0 + 5 = 9 - 4 = 5 \cdot 1 = \frac{5}{2}\frac{2}{1} = \frac{5\pi}{\pi}$
2. $2.3 = 2.2 + 0.1 = \frac{2.3}{1} = (2 \cdot 1) + (3 \cdot \frac{1}{10}) = 2 + 0.3 = \frac{23}{10} = \frac{230}{100} = 2.3 + (3 - 3) = \frac{2.3}{1} \cdot \frac{10}{10} = 3 - 0.7$

Of course, *many* other names are possible.

EXERCISE 7.

1. '$-4 - (-3)$' is read as '*negative* 4 *minus negative* 3'
2. On many calculators, the least positive number that can be represented is:

$$L := 1.0000000 \times 10^{-99}$$

   (The symbol ':=' means 'equal, by definition'.)

   When this number is divided by 2, a less positive real number is obtained: since the calculator cannot represent this less positive number, it calls the result '0'. Indeed, the author's calculator gives the answer 0 for $\frac{L}{2}$. Of course, the correct answer is 0.5×10^{-99}.

3. On many calculators, the greatest positive number that can be represented is:

$$G := 9.9999999 \times 10^{99}$$

If 1 is added to G, the calculator is unable to represent this greater number, so it is stored as simply G. Then, when G is subtracted, the result is 0. Indeed, the authors's calculator gives the answer 0 for $(G+1)-G$. Of course, the correct answer is 1 .

EXERCISE 8.

1. The word 'small' refers to SIZE, whereas the word 'less' refers to POSITION.

   One way to measure the 'size' of a number is by using its distance from zero; the absolute value gives us this measure. Note that $|-5| = 5$ and $|3| = 3$. Relative to this measure of size, -5 is NOT 'smaller than' 3 .
2. The word 'big' refers to SIZE, whereas the word 'greater' refers to POSITION.

   Let $n = -3$ and $m = -5$. Then, $-3 > -5$ is true, since -3 lies to the right of -5 on the number line. However, $|-3| = 3$ and $|-5| = 5$, so, relative to the absolute value measure of size, -3 is NOT 'bigger than' -5 .
3. a) $-3 < 2$ is true

   b) To compare fractions, get common denominators. The least common multiple of the original denominators 5 and 7 is 35, so:

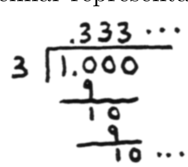
   $$\frac{2}{5} = \frac{2}{5} \cdot \frac{7}{7} = \frac{14}{35} \quad \text{and} \quad \frac{3}{7} = \frac{3}{7} \cdot \frac{5}{5} = \frac{15}{35}$$

   The problem is therefore rephrased as $\frac{14}{35} > \frac{15}{35}$, which is false.

   c) $-3 > -7$ is true, since -3 lies to the right of -7 on the number line

END-OF-SECTION EXERCISES:

1. '$\frac{1}{3}$' is an expression
2. 'π' is an expression; it is the symbol for the irrational number 'pi', which gives the ratio of the circumference to the diameter of *any* circle
3. '$\frac{1}{3} = 0.\overline{3}$' is a true sentence. The bar over the 3 indicates that the overlined pattern continues ad infinitum: $0.\overline{3} = 0.333333333\ldots$ This decimal representation for the number $\frac{1}{3}$ is obtained by long division:



4. '$\frac{1}{3} = 0.3\overline{3}$' is a true sentence. The name '$0.3\overline{3}$' is just another name for '$0.\overline{3}$'.
5. '$\frac{1}{3} = 0.33$' is false. The numbers $\frac{1}{3}$ and 0.33 are different points on the number line; however, they are very close to each other.
6. '$\frac{1}{3} = 0.33333$' is a false sentence. However, the numbers $\frac{1}{3}$ and 0.33333 are extremely close on the number line. Can you give a number that is yet *closer* to $\frac{1}{3}$?
7. '$\frac{1}{3} \approx 0.33$' is true. The symbol '$\approx$' means '*is approximately equal to*'.
8. '$\frac{1}{3} \approx 0.333333$' is a true sentence
9. '$x^2 > 0$' is a sentence that is sometimes true/sometimes false. Indeed, it is true for all nonzero real numbers, and false when x is 0 .
10. '$y^2 > 0$' is a sentence that is sometimes true/sometimes false; see (9)
11. '$x^2 \geq 0$' is a true sentence, for all real numbers x
12. '$y^2 \geq 0$' is a true sentence, for all real numbers y
13. '$(-3)(-5)$' is an expression; this is an alternate name for the number 15

14. '$(-3)+(-5)$' is an expression; this is an alternate name for the number -8
15. '$-5 < -3$' is a true sentence, since -5 lies to the left of -3 on the number line
16. '$-3 < -5$' is a false sentence, since -3 does not lie to the left of -5 on the number line
17. '$|t| > 0$' is a sentence that is sometimes true/sometimes false. Indeed, it is true for all nonzero real numbers t, and false if t is 0.
18. '$|x| > 0$'; same answer as (17) (with 't' replaced by 'x')
19. '$|t| \geq 0$' is a true sentence, for all real numbers t
20. '$|x| \geq 0$'; same answer as (19) (with 't' replaced by 'x')
21. '$|t| < 0$' is a false sentence, for all real numbers t
22. '$|x| < 0$' is a false sentence, for all real numbers x
23. '$|3-\pi| = \pi - 3$' is a true sentence. Since $\pi > 3$, the number $3-\pi$ is negative, and hence its distance from 0 is $-(3-\pi) = \pi - 3$.
24. '$|\pi - 3| = \pi - 3$' is a true sentence
25. '$|t| = t$' is a sentence that is sometimes true/sometimes false. Indeed, it is true for all $t \geq 0$, but false for $t < 0$.
26. '$|t| = -t$' is a sentence that is sometimes true/sometimes false. Indeed, it is true for all $t \leq 0$, but false for $t > 0$.
27. '$\frac{x}{y} \div \frac{z}{w}$' is an expression
28. '$\frac{x}{y} \cdot \frac{w}{z}$' is an expression
29. '$\frac{x}{y} \div \frac{z}{w} = \frac{xw}{yz}$' is a true sentence, for all values of x, y, z and w that are allowable. Note that y, w and z are not allowed to equal 0, since this would produce division by zero. This formula expresses the correct way to divide fractions.
30. '$\frac{x}{y} \cdot \frac{w}{z} = \frac{xw}{yz}$' is a true sentence, for all allowable values of x, y, z and w. Note that y and z are not allowed to equal zero. This formula expresses the correct way to multiply fractions.
31. '$a(bc) = (ab)c$' is a true sentence, for all real numbers a, b and c. This property is referred to as the Associative Law of Multiplication.
32. '$a + (b+c) = (a+b)+c$' is a true sentence, for all real numbers a, b and c. This property is referred to as the Associative Law of Addition.
33. '$3x^2 = (3x)^2$' is a sentence that is sometimes true/sometimes false. Indeed, it is true only when $x = 0$, and false otherwise.
34. '$(2\cdot 3)^2 = = 2\cdot 3^2$' is a false sentence, since $(2\cdot 3)^2 = 6^2 = 36$ and $2\cdot 3^2 = 2\cdot 9 = 18$
35. '$\sqrt{(-3)^2} = -3$' is a false sentence, since $\sqrt{(-3)^2} = \sqrt{9} = 3$
36. '$\sqrt{(-3)^2} = 3$' is a true sentence
37. This is the Commutative Property of Addition. In English, the word 'commute' means, roughly, 'to change places'. (For example, if a person 'commutes' to work, they travel to work, and then back home again.) All 'commutative' laws have to do with the fact that *changing places does not change the final result.*
38. This is the Commutative Property of Multiplication. See the discussion in (37).
39. This is the Distributive Law (of multiplication over addition). All 'distributive' laws have to do with how two operations interact; here, we are seeing how *multiplication* interacts with *addition.*
40. This is the Associative Property of Multiplication. See the statement of (43) for additional discussion.
41. If $x = 1$ and $y = 3$, we have $1 - 3 = 1 + (-3) = -2$. If $x = 1$ and $y = -3$, we have $1 - (-3) = 1 + (-(-3)) = 1 + 3 = 4$.

42. Division is just a special kind of multiplication, since for all real numbers x and y (y nonzero) we have:

$$\frac{x}{y} = x \cdot \frac{1}{y}$$

Thus, division by y is the same as multiplication by the reciprocal of y.

43. The expression xyz is not ambiguous; if one person computes this as $(xy)z$ and another as $x(yz)$, the same results are obtained.

44. The Associative Law of Addition allows us to write things like $a + b + c$ without ambiguity. If one person computes this as $(a + b) + c$ and another as $a + (b + c)$, the same results are obtained.