CHAPTER 7. THE DEFINITE INTEGRAL

Section 7.1 Using Antiderivatives to find Area

Quick Quiz:
1. The Max-Min Theorem guarantees numbers \( m \in [x, x + h] \) and \( M \in [x, x + h] \) for which \( f(m) \) is the minimum value of \( f \) on \([x, x + h]\), and \( f(M) \) is the maximum value of \( f \) on \([x, x + h]\).
2. If \( f \) is continuous at \( a \), then as \( x \to a \), it must be that \( f(x) \to f(a) \).
3. Any sketch where \( f \) IS defined at \( a \), but \( f \) is NOT continuous at \( a \), will work!
4. \( F(x) = x^3 \) is an antiderivative of \( f(x) = 3x^2 \). Then, the desired area is given by: \( F(2) - F(0) = 2^3 - 0^3 = 8 \)
5. The desired area is given by: \( F(d) - F(c) \)

END-OF-SECTION EXERCISES:
1. approximation by a triangle: \( \frac{1}{2}(1)(e - 1) \approx 0.86 \)
   actual area: Using integration by parts, an antiderivative of \( f(x) = \ln x \) is \( F(x) = x \ln x - x \). Then:
   \[
   F(e) - F(1) = (e \ln e - e) - (1 \ln 1 - 1) = (e - e) - (0 - 1) = 1
   \]
3. approximation by a trapezoid: \( \frac{1}{2}(4 - 1)(1 + 2) = \frac{1}{2}(9) = \frac{9}{2} = 4.5 \)
   actual area: An antiderivative of \( f(x) = \sqrt{x} = x^{1/2} \) is \( F(x) = \frac{2}{3}x^{3/2} = \frac{2}{3}\sqrt{x^3} \). Then:
   \[
   F(4) - F(1) = \frac{2}{3}\sqrt{4^3} - \frac{2}{3}\sqrt{1^3} = \frac{2}{3}(8) - \frac{2}{3}(1) = \frac{2}{3}(7) = \frac{14}{3} \approx 4.67
   \]

Section 7.2 The Definite Integral

Quick Quiz:
1. The indefinite integral \( \int f(x) \, dx \) gives all the antiderivatives of the function \( f \); by the Fundamental Theorem of Integral Calculus, if just one of these antiderivatives is known, then the definite integral \( \int_a^b f(x) \, dx \) can be computed!
2. See page 409.
3. The notation \( F(x) \big|_a^b \) means \( F(b) - F(a) \).
4. \( \int_{-1}^{1} x^2 \, dx = \left[ \frac{x^3}{3} \right]_{-1}^{1} = \frac{1}{3}(2^3 - (-1)^3) = \frac{1}{3}(8 - (-1)) = \frac{1}{3}(9) = 3 \)
5. \( \int_{-1}^{1} x^3 \, dx = \left[ \frac{x^4}{4} \right]_{-1}^{1} = \frac{1}{4}(1^4 - (-1)^4) = 0 \). On the interval \([-1, 1]\), there is the same amount of area above the graph of \( y = x^3 \), as there is below.

END-OF-SECTION EXERCISES:
1. \( \frac{48}{5} \)
3. \(-6\)
5. \( \frac{1}{3} \ln 2 \)
7. \( 1 + e^2 \)
9. The desired area is: \( \frac{19}{24} + \frac{81}{8} = \frac{131}{24} \)
Section 7.3 The Definite Integral as the Limit of Riemann Sums

Quick Quiz:
1. A partition of an interval \([a, b]\) is a finite set of points from \([a, b]\) that includes both \(a\) and \(b\).
2. The length of the longest subinterval must be \(\frac{1}{2}\):
   \[
P_1 = \{1, 1.5, 2, 2.5, 3\}
   \]
   \[
P_2 = \{1, 1.3, 1.5, 2, 2.5, 3\}
   \]
3. There is NOT a unique Riemann sum for \(f\) corresponding to this partition; any number \(x^*_1\) may be chosen from the subinterval \([0, 1)\); any number \(x^*_2\) may be chosen from the second subinterval \([1, 2)\), etc.
4. Think of a rectangle with ‘width’ \(dx\) and ‘height’ \(f(x)\), where \(x\) is a number between \(a\) and \(b\).

END-OF-SECTION EXERCISES:
1. EXP
3. SENTENCE; TRUE
5. SENTENCE; TRUE
7. SENTENCE; TRUE
9. SENTENCE; TRUE

Section 7.4 The Substitution Technique applied to Definite Integrals

Quick Quiz:
1. \[
\int (2x - 1)^3 \, dx = \frac{1}{2} \int u^3 \, du = \frac{1}{8} x^4 + C ; \quad \mu = 2x - 1
\]
2. \[
\int_0^{\sqrt{2}} (2x - 1)^3 \, dx = \frac{1}{8} (2x - 1)^4 \bigg|_0^{\sqrt{2}} = \frac{1}{8} (0 - 1) = -\frac{1}{8}
\]
3. \[
\int_1^e \ln x \, dx = x \ln x \bigg|_1^e - \int_1^e x \cdot \frac{1}{x} \, dx
= (e \ln e - 1 \ln 1) - e \bigg|_1^e
= e - (e - 1) = 1
\]

END-OF-SECTION EXERCISES:
1. \(0\)
3. \(\approx 0.024\)
5. \(\approx 1.931\)

Section 7.5 The Area Between Two Curves

Quick Quiz:
1. \(\int_0^d (g(x) - f(x)) \, dx\)
2. The \(x\)-axis is described by \(y = 0\). The intersection points are found by:
   \[-x^2 + 1 = 0 \iff x^2 = 1 \iff x = \pm 1\]

   Using symmetry, the desired area is:
   \[
   2 \int_0^1 (-x^2 + 1) \, dx = (x - \frac{x^3}{3}) \bigg|_0^1 - \frac{1}{3} = \frac{2}{3}
   \]
3. A quick sketch shows that the phrase IS ambiguous; there are two regions with the indicated boundaries. Which is desired? Or, are both desired?
4. \[ \int_0^1 (e^x - (-x)) \, dx = \int_0^1 (e^x + x) \, dx = (e^x + \frac{x^2}{2}) \bigg|_0^1 = (e + \frac{1}{2}) - e^0 = e + \frac{1}{2} - 1 = e - \frac{1}{2} \]

END-OF-SECTION EXERCISES:

1. \( \frac{2}{15} \)

2. \( \frac{32}{3} \)

3. \( \approx 2.438 \)

4. \( 20 \frac{1}{4} \)

Section 7.6 Finding the Volume of a Solid of Revolution—Disks

Quick Quiz:

1. Revolve \( x = r \) about the \( y \)-axis; or revolve \( y = r \) about the \( x \)-axis.

2. Revolve \( y = -\frac{h}{r}x + h \) about the \( y \)-axis; or revolve \( y = \frac{h}{r}x \) about the \( y \)-axis. (There are other correct answers.)

3. \[ \int_0^1 \pi (x^2)^2 \, dx = \pi \frac{x^5}{5} \bigg|_0^1 = \frac{\pi}{5} (1 - 0) = \frac{\pi}{5} \]

4. Intersection points of \( y = x^2 \) and \( y = 1 \): \( x^2 = 1 \iff x = \pm 1 \)

   Also: \( y = x^2 \iff x = \pm \sqrt{y} \)

   A typical ‘slice’ at a distance \( y \) has volume \( \pi (\sqrt{y})^2 \, dy \). The desired volume is:

   \[ \int_0^1 \pi (\sqrt{y})^2 \, dy = \int_0^1 \pi y \, dy = \frac{\pi y^2}{2} \bigg|_0^1 = \frac{\pi}{2} (1 - 0) = \frac{\pi}{2} \]
END-OF-SECTION EXERCISES:

1. $\frac{4\pi}{3}$

3. $\frac{\pi}{2}$

5. $8\pi$

7. $\frac{128\pi}{5}$

9. $\frac{8\pi}{3}$

11. $\frac{\pi}{4}$

Section 7.7 Finding the Volume of a Solid of Revolution—Shells

Quick Quiz:

1. ‘Cut’ the shell and unroll it; the volume is:

$$2\pi r \cdot h \cdot dx$$

2. 

$$\int_{0}^{2} 2\pi x(x) \, dx = 2\pi \frac{x^3}{3} \bigg|_{0}^{2} = \frac{2\pi}{3} (8 - 0) = \frac{16\pi}{3}$$

3. To use horizontal disks would require disks ‘with holes’. Thus, in this case, shells are easier to use.

END-OF-SECTION EXERCISES:

1. $\frac{4\pi}{3}$

3. $2\pi$