CHAPTER 4. THE DERIVATIVE

Section 4.1 Tangent Lines

Quick Quiz:
1. Let \( f(x) = x \). Then:
   \[
   \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{(2 + h) - 2}{h} = \lim_{h \to 0} \frac{h}{h} = 1
   \]
   Thus, as expected, the slope of the tangent line to \( f \) at the point \((2, 2)\) is 1.
2. The dummy variable is \( h \). Using the dummy variable \( t \), the limit can be rewritten as:
   \[
   \lim_{t \to 0} \frac{f(x + t) - f(x)}{t}
   \]
3. In the limit, \( x \) represents the \( x \)-value of a point where the slope of the tangent line is desired.
4. In the limit, the difference quotient \( \frac{f(x+h) - f(x)}{h} \) represents the slope of a secant line through the points \((x, f(x))\) and \((x + h, f(x + h))\). This secant line is being used as an approximation to the tangent line at the point \((x, f(x))\).
5. The function \( f \) is graphed below. Since \( f \) is only defined to the right of 0, the limit is actually a right-hand limit:
   \[
   \lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^2 - 0}{h} = \lim_{h \to 0^+} h = 0
   \]
   The slope of the tangent line at the point \((0, 0)\) is 0.

End-of-Section Exercises:
1. EXP
3. SEN; CONDITIONAL
5. SEN; TRUE
7. \( g(0.1) = \frac{f(x+0.1) - f(x)}{0.1} \); \( g(\Delta x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} \)
9. \( h \in \mathcal{D}(g) \iff (h \neq 0 \text{ and } x + h \in \mathcal{D}(f)) \)
11. When \( \lim_{h \to 0} g(h) \) exists, it tells the slope of the tangent line to the graph of \( f \) at the point \((x, f(x))\).

Section 4.2 The Derivative

Quick Quiz:
1. When the limit exists:
   \[
   f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
   \]
2. \( f' \) is the derivative function; \( f'(x) \) is a particular output of this function, when the input is \( x \).
3. \( A - B = (0, 2) \cup (2, 4); B - A = \{4\} \)
4. $\mathcal{D}(f') = \mathbb{R} - \{1\}$; its graph is:

![Graph of f'](image)

5. TRUE. If the limit $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ exists, then, in particular, $f$ must be defined at $x$ (so that $f(x)$ makes sense).

End-of-Section Exercises:

1. The graph of $f$ is shown below. Here, $\mathcal{D}(f) = \mathbb{R}$.

![Graph of f](image)

When $x > 2$, the slopes of the tangent lines equal 1.
When $x < 2$, the slopes of the tangent lines equal $-1$.
There is no tangent line at $x = 2$.
The graph of $f'$ is shown at right. Here, $\mathcal{D}(f') = \mathbb{R} - \{2\}$.

3. The graph of $f$ is shown below. Here, $\mathcal{D}(f) = \mathbb{R}$.

![Graph of f](image)

When $x > 1$, the slopes of the tangent lines equal 2.
When $x < 1$, the slopes of the tangent lines equal $2x$ (as per an example in the text).
There is no tangent line at $x = 1$.
The graph of $f'$ is shown at right. Here, $\mathcal{D}(f') = \mathbb{R} - \{1\}$.
5. Note that \( f(2) = \frac{1}{2 - 1} = 1 \). Then:

\[
\lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{\frac{1}{(2 + h) - 1} - 1}{h}
\]

\[
= \lim_{h \to 0} \frac{\frac{1}{1 + h} - \frac{1}{1 + h}}{h}
\]

\[
= \lim_{h \to 0} \frac{1 - (1 + h)}{h(1 + h)}
\]

\[
= \lim_{h \to 0} \frac{-1}{1 + h} = -1
\]

Thus, \( f'(2) = -1 \). That is, the slope of the tangent line to the graph of \( f \) at the point \((2, 1)\) is \(-1\).

7. \( y - 9 = 6(x - 3) \)

9. \( y = 1 \)

Section 4.3 Some Very Basic Differentiation Formulas

Quick Quiz:
1. \( f(x) = x^{3/2}; \ f'(x) = \frac{3}{2}x^{-1/2} = \frac{3}{2\sqrt{x}} \). In Leibniz notation: \( \frac{df}{dx} = \frac{3}{2\sqrt{x}} \)
2. TRUE. The derivative of a constant equals zero.
3. \( y' = 3x^2; \) the slope of the tangent line at \( x = 2 \) is \( y'(2) = 3(2^2) = 12 \). TRUE.
4. \[(a - b)^4 = (a + (-b))^4 = (1)a^4 + (4)a^3(-b) + (6)a^2(-b)^2 + (4)a(-b)^3 + (1)(-b)^4
\]

\[= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4\]

5. \( g'(x) = e^x + \frac{1}{x} \)

End-of-Section Exercises:
1. Multiply out, differentiate term-by-term, and simplify: \( f'(x) = 6(2x + 1)^2 \)
3. \( h'(x) = \begin{cases} 6x - 2 & \text{for } x \geq 1 \\ 4 & \text{for } x < 1 \end{cases} \)

\( \mathcal{D}(h) = \mathcal{D}(h') = \mathbb{R} \)

Section 4.4 Instantaneous Rates of Change

Quick Quiz:
1. \( \frac{f(2) - f(1)}{2 - 1} = \frac{3^3 - 1^3}{1} = 8 - 1 = 7 \); this number represents the slope of the secant line through the points \((1, 1^3)\) and \((2, 2^3)\)
2. \( f'(x) = 3x^2; \ f'(1) = 3(1) = 3 \). This number represents the slope of the tangent line at the point \((1, 1^3)\).
3. less than; once we move to the right of \( x = 1 \), the rates of change increase
4. One correct sketch is given:

5. Since \( f \) is not continuous at \( x = 1 \), \( f \) is not differentiable at \( x = 1 \).
End-of-Section Exercises:
In all cases, the ‘predicted value’ for $f(x_2)$ from known information at $x_1$ is given by

$$f(x_2) \approx f(x_1) + (\Delta x)(f'(x_1)),$$

where $\Delta x_2 = x_2 - x_1$.

1. Here, $\Delta x = 2 - 1 = 1$; $f(2) \approx 3 + (1)(2) = 5$
3. Here, $\Delta x = 4 - 3 = 1$; $f(4) \approx -1 + (1)(5) = 4$

**Section 4.5 The Chain Rule (Differentiating Composite Functions)**

Quick Quiz:
1. See page 231. The Chain Rule tells us how to differentiate composite functions.
2. $f'(x) = 7\sqrt{2}(1-x)^6(-1) = -7\sqrt{2}(1-x)^6$
3. $\frac{dy}{dt} = \frac{du}{dx} \cdot \frac{dw}{dx} \cdot \frac{dv}{dx} \cdot \frac{du}{dt}$
4. ... tells us that to find out how fast $f \circ g$ changes with respect to $x$, we find out how fast $f$ changes with respect to $g(x)$, and multiply by how fast $g$ changes with respect to $x$
5. $f(x) = \frac{1}{3} \ln(2x+1), \quad f'(x) = \frac{1}{3} \cdot \frac{1}{2x+1} \cdot 2 = \frac{2}{3(2x+1)}$

End-of-Section Exercises:

1. $f'(x) = \frac{-e^x}{\sqrt{(e^x - 1)^3}} + 1$
3. $\frac{dy}{dx} = 3e^{3x}$
5. $y' = 33(3t - 4)^{10}$
7. $g'(t) = \frac{2t + 1}{\sqrt{t^2 + 2t + 1}}$
9. $f'(y) = -7e^{-y} + \frac{1}{y}$
11. $\frac{dy}{dx} = \frac{3}{x} \left( \ln x \right)^2$
13. $\frac{dy}{dt} = \frac{2\sqrt{t-1} + 1}{2\sqrt{t-1}(t + \sqrt{t-1})^2}$

**Section 4.6 Differentiating Products and Quotients**

Quick Quiz:
1. See page 239.
2. See page 244.
3. $f'(x) = x \cdot 5(x+1)^4(1) + (1)(x+1)^5$
4. Using the Quotient Rule:

$$f'(x) = \frac{e^{2x}(2) - (2x+1) \cdot 2e^{2x}}{(e^{2x})^2}$$

$$= \frac{2e^{2x}(1 - (2x+1))}{e^{4x}}$$

$$= -\frac{4x e^{2x}}{e^{4x}}$$
5. Using a ‘generalized’ product rule:

\[ y' = (1)(x + 1)(x^2 + 1) + x(1)(x^2 + 1) + x(x + 1)(2x) \]

End-of-Section Exercises:
1. \[ y' = 2(2 - x)^2(1 - 2x) \]
   \[ y(0) = 0, \quad y(t^2) = t^2(2 - t^2)^3 \]
   \[ y'(0) = 8, \quad y'(t) = 2(2 - t)^2(1 - 2t) \]
3. \[ f'(x) = e^x \left( \frac{1}{x} + \ln x \right) \]
   \[ D(f) = (0, \infty), \quad D(f') = (0, \infty) \]
   \[ f'(e) = e^e \left( \frac{1}{e} + x \right), \quad f'(e^2) = e^{e^2} \left( \frac{1}{e^2} + 2 \right) \]
5. \[ g'(x) = e^{x+e^x} \]
   \[ \lim_{x \to 0} g(x) = e, \quad \lim_{x \to 0} g'(x) = e \]
   \[ D(g) = \mathbb{R}, \quad g(g'(0))) = e^{e+e^e} \]
7. \[ h'(x) = \frac{x}{x+1}; \text{ the tangent line is horizontal, and has equation } y = 0 \]
9. \[ f'(x) = 4e^{2x}(2x + 1)^6(x + 4); \text{ the tangent line has equation } y = 16x + 1 \]
11. \[ h(t) = \frac{-12e}{(3t-1)^2}; \text{ the tangent line has equation } y - e = -12e(t - \frac{2}{3}) \]
13. \[ y' = 0 \iff (x = 3 \text{ or } x = -1 \text{ or } x = \frac{1}{2} \text{ or } x = \frac{3 \pm \sqrt{17}}{2}) \]

Section 4.7 Higher Order Derivatives

Quick Quiz:
1. The ‘higher derivatives’ of a function \( f \) are the derivatives of the form \( f^{(n)}(x) \) for \( n \geq 2 \). That is, the second derivative, third derivative, fourth derivative, etc., are the ‘higher derivatives’ of \( f \).
2. prime notation: \( f''(x) \)
   Leibniz notation: \( \frac{d^2 f}{dx^2}(x) \)
3. \[ \sum_{i=1}^{3} i^{i+1} = 1^{1+1} + 2^{2+1} + 3^{3+1} = 1 + 8 + 81 = 90 \]
4. \[ 10 \cdot 9 \cdot 7 \cdot 6 = 10 \cdot 9 \cdot 7 \cdot 6 \cdot \frac{5!}{5!} = \frac{10!}{5!} \]
5. \[ \frac{d}{dx} \sum_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} f_i'(x) \]

End-of-Section Exercises:
1. SEN; TRUE
3. EXP
Section 4.8 Implicit Differentiation (Optional)

Quick Quiz:
1. \[
\frac{d}{dx}(xy^2) = \frac{d}{dx}(2)
\]
\[x(2y^1)\frac{dy}{dx} + (1)y^2 = 0\]
\[\frac{dy}{dx} = \frac{-y^2}{2xy}\]

2. For \(x > 0\):
\[
\ln y = \ln(x^{2x}) = 2x \ln x
\]
\[\frac{1}{y} \frac{dy}{dx} = (2x) \frac{1}{x} + (2) \ln x = 2 + 2 \ln x = 2(1 + \ln x)\]
\[\frac{dy}{dx} = y \cdot 2(1 + \ln x) = 2x^{2x}(1 + \ln x)\]

3. Put the equation in standard form, by completing the square:
\[x^2 - 2x + y^2 = 8 \iff (x^2 - 2x + (\frac{-2}{2})^2) + y^2 = 8 + 1\]
\[\iff (x - 1)^2 + (y - 0)^2 = 3^2\]

The equation graphs as the circle centered at \((1, 0)\) with radius 3.

4. There are many possible correct answers. Here are two:
- \(y\) given explicitly in terms of \(x\): \(y = x^2 + 2x + 1\)
- \(y\) given implicitly in terms of \(x\): \(xy^2 = x + y\)

End-of-Section Exercises:
1. The graph is the circle centered at \((-2, 1)\) with radius 1.
   \(y\) is NOT locally a function of \(x\) at the points \((-1, 1)\) and \((-3, 1)\). (There are vertical tangent lines here.)
   The equation of the tangent line at the point \((-2, 2)\) is \(y = 2\).
   The equation of the tangent line at the point \((-1, 1)\) is \(x = -1\).

3. The graph is the circle centered at \((-2, 1)\) with radius 2.
   \(y\) is NOT locally a function of \(x\) at the points \((0, 1)\) and \((-4, 1)\); there are vertical tangent lines here.
   The equation of the tangent line at the point \((-1, 1 + \sqrt{3})\) is:
\[y - (1 + \sqrt{3}) = -\frac{1}{\sqrt{3}}(x - (-1))\]
Section 4.9 The Mean Value Theorem

Quick Quiz:
1. See page 266.
2. The word ‘mean’ refers to ‘average’; the Mean Value Theorem guarantees (under certain hypotheses) a place in an interval $(a, b)$ where the instantaneous rate of change is the same as the average rate of change over the entire interval.
3. The average rate of change of $f$ on the interval $[1, 3]$ is:
   \[
   f(3) - f(1) \over 3 - 1 = 27 - 1 \over 2 = 13
   \]
   The instantaneous rates of change are given by $f'(x) = 3x^2$. We seek $c \in (1, 3)$ for which $f'(c) = 13$:
   \[
   f'(c) = 13 \iff 3c^2 = 13 \\
   \iff c^2 = \frac{13}{3} \\
   \iff c = \pm \sqrt{\frac{13}{3}}
   \]
   Choosing the value of $c$ in the desired interval, we get $c = \sqrt{\frac{13}{3}}$.
4. If $f$ WERE continuous on $[a, b]$, then there would have to be (by the MVT) a number $c \in (a, b)$ with $f'(c) = \frac{f(b) - f(a)}{b - a}$. Thus, it must be that $f$ is NOT continuous on $[a, b]$; that is, $f$ ‘goes bad’ at (at least one) endpoint.
5. If $f$ WERE differentiable on $(a, b)$, then the MVT would guarantee that there must be $c \in (a, b)$ with $f'(c)$ equal to the average rate of change of $f$ over $[a, b]$. Therefore, we can conclude that $f$ is NOT differentiable on $(a, b)$. That is, there is at least one value of $x$ in the interval $(a, b)$ where $f'(x)$ does not exist.

End-of-Section Exercises:
1. The limit gives the slope of the tangent line to the graph of $f$ at the point $(x, f(x))$, whenever the tangent line exists and is non-vertical.
3. There is a tangent line to the graph of $f$ when $x = 2$, and its slope is 4.
5. Let $f(x) = -x^2$. Then:
   \[
   f'(x) := \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{-(x + h)^2 - (-x^2)}{h} \\
   = \lim_{h \to 0} \frac{-(x^2 + 2xh + h^2) + x^2}{h} = \lim_{h \to 0} \frac{h(-2x - h)}{h} \\
   = \lim_{h \to 0} (-2x - h) = -2x
   \]
7. Put a ‘kink’ in the graph when $x = 3$.
9. 
   \[
   f'(x) = e^{2x} \ln(2 - x) + 2xe^{2x} \ln(2 - x) - \frac{x e^{2x}}{2 - x}
   \]
   \[
   \mathcal{D}(f) = (-\infty, 2), \quad \mathcal{D}(f') = (-\infty, 2)
   \]
   The tangent line when $x = 0$ has equation $y = (\ln 2)x$. 