

CHAPTER 2. FUNCTIONS

Section 2.1 Functions and Function Notation

Quick Quiz:

1. In the graph shown, y is a function of x , because for every x , there is a unique y . That is, the graph passes a vertical line test.

However, x is not a function of y . It is NOT true that for every y , there is a unique x . That is, the graph does NOT pass a horizontal line test.

2.

$$x^2 - y + 1 = 0 \iff y = x^2 + 1$$

For every value of x , there is a unique value of y . Thus, y is a function of x .

3.

$$\begin{aligned}x^2 - y + 1 = 0 &\iff x^2 = y - 1 \\ &\iff x = \pm\sqrt{y - 1}\end{aligned}$$

For each allowable y -value, there are *two* x -values. Therefore, x is NOT a function of y .

4. Calling the function f : $f(x) = (\frac{x}{2} - 3)^2$

5. $g(-1) = 2(-1)^2 - 1 = 2 - 1 = 1$

$$g(x^2) = 2(x^2)^2 - 1 = 2x^4 - 1$$

End-of-Section Exercises:

1. $f(0) = 0^3 - 1 = -1$

$$f(1) = 1^3 - 1 = 0$$

$$f(-1) = (-1)^3 - 1 = -1 - 1 = -2$$

$$f(t) = t^3 - 1$$

$$f(f(2)); \text{ first find } f(2): f(2) = 2^3 - 1 = 7; \text{ then, } f(f(2)) = f(7) = 7^3 - 1 = 342$$

3. $f(-2) = |-2| = 2$

$$f(t) = |t| = \begin{cases} t & \text{if } t \geq 0 \\ -t & \text{if } t < 0 \end{cases}$$

$$f(-t) = |-t| = |t|$$

$$f(x^2) = |x^2| = |x|^2$$

5. $h(-x) = \frac{1}{-x} = -\frac{1}{x}$

$$h(h(x)) = h\left(\frac{1}{x}\right) = \frac{1}{1/x} = x$$

$$h\left(\frac{1}{x}\right) = \frac{1}{1/x} = x$$

$$h(x + \Delta x) = \frac{1}{x + \Delta x}$$

$$h(|x|) = \frac{1}{|x|} = \left|\frac{1}{x}\right|$$

7. $h(1, 1) = 1^2 + 1^2 - 1 = 1$

$$h(x, x) = x^2 + x^2 - 1 = 2x^2 - 1$$

$$h(y, x) = y^2 + x^2 - 1 = h(x, y)$$

$$h(x + \Delta x, y + \Delta y) = (x + \Delta x)^2 + (y + \Delta y)^2 - 1$$

Section 2.2 Graphs of Functions

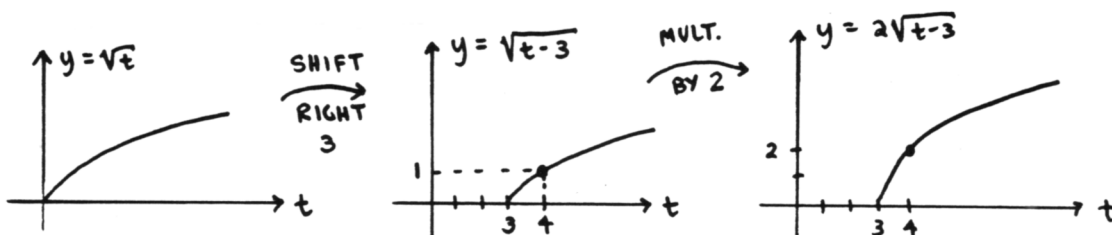
Quick Quiz:

1.

$$\begin{aligned} \mathcal{D}(f) &= \{x \mid 2x - 1 \geq 0 \text{ and } x^2 - 9 \neq 0\} \\ &= \{x \mid 2x \geq 1 \text{ and } x^2 \neq 9\} \\ &= \{x \mid x \geq \frac{1}{2} \text{ and } |x| \neq 3\} \\ &= \{x \mid x \geq \frac{1}{2} \text{ and } x \neq 3 \text{ and } x \neq -3\} \\ &= \{x \mid x \geq \frac{1}{2} \text{ and } x \neq 3\} \end{aligned}$$

Note that if $x \geq \frac{1}{2}$, then automatically, x is not equal to -3 .

- TRUE. The order that elements are listed in a set is unimportant. In this sentence, the '=' sign is being used for equality of SETS.
- By definition, the *graph of f* is the set of points $\{(x, f(x)) \mid x \in \mathcal{D}(f)\}$. More precisely, the graph usually refers to a (partial) *picture* of this set of points, in the xy -plane.
- $\mathcal{D}(f) = [3, \infty)$; the graph is 'built up' below:



- $P(-1) = (-1)^4 - 2(-1)^2 + 1 = 1 - 2 + 1 = 0$; therefore -1 is a root of P . Long division by $x - (-1) = x + 1$ yields:

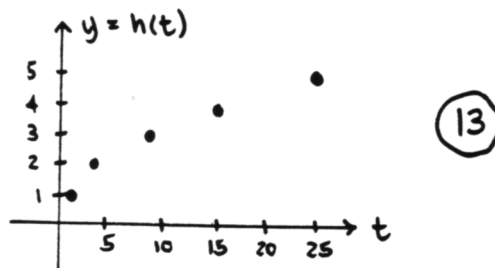
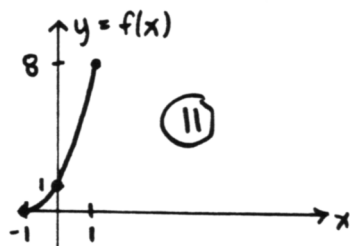
$$\begin{array}{r} x^3 - x^2 - x + 1 \\ x+1 \overline{) x^4 - 2x^2 + 1} \\ \underline{-(x^4 + x^3)} \\ -x^3 - 2x^2 + 1 \\ \underline{-(-x^3 - x^2)} \\ -x^2 + 1 \\ \underline{-(-x^2 - x)} \\ x + 1 \\ \underline{x + 1} \\ 0 \end{array}$$

Therefore: $P(x) = (x + 1)(x^3 - x^2 - x + 1)$

End-Of-Section Exercises:

- EXP; this is a set.
- SEN; TRUE. The component sentences ' $x \geq 2$ and $x \neq 1$ ' and ' $x \geq 2$ ' always have the same truth values. Both are true on $[2, \infty)$ and false elsewhere.
- SEN; TRUE
- SEN; TRUE. Both sets are equal to $\{3\}$.
- SEN; this is TRUE (by definition), providing g is a function of one variable.

The graphs requested in problems 11 and 13 are given below:

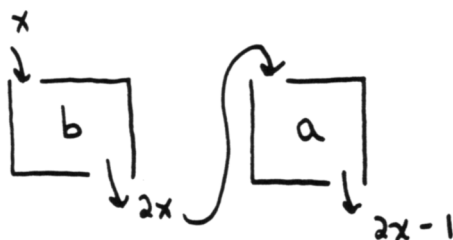


Section 2.3 Composite Functions

Quick Quiz:

1. $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$. If $A = [1, 3]$ and $B = \{1, 2, 3\}$, then $A \cap B = \{1, 2\}$ since the only elements that are in *both* A and B are 1 and 2.
2. The sentence ' $[1, 3] \subset \{1, 3\}$ ' is FALSE. For example, $2 \in [1, 3]$, but $2 \notin \{1, 3\}$.
The sentence ' $\{1, 3\} \subset [1, 3]$ ' is TRUE.
The sentence 'For all sets A and B , $A \cap B \subset A$ ' is TRUE. Everything that is in BOTH A and B , is also in A .
3. $(f + g)(x) := f(x) + g(x)$
 $\mathcal{D}(f + g) = \{x \mid x \in \mathcal{D}(f) \text{ and } x \in \mathcal{D}(g)\} = \mathcal{D}(f) \cap \mathcal{D}(g)$
4. The function f takes an input x , multiplies by 2, then subtracts 1. Define $b(x) = 2x$ and $a(x) = x - 1$; then:

$$\begin{aligned}
 (a \circ b)(x) &:= a(b(x)) \\
 &= a(2x) \\
 &= 2x - 1 \\
 &:= f(x)
 \end{aligned}$$



5. $\mathcal{R}(f) = \{1, -1\}$. The only two output values taken on by f are 1 and -1 .

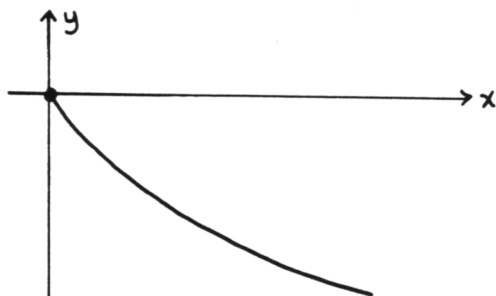
End-Of-Section Exercises:

1. EXP; $A \cup B$ is a set.
3. SEN; CONDITIONAL. The truth of the sentence ' $A \subset B$ ' depends upon the sets A and B .
5. SEN; CONDITIONAL. The sentence ' $\mathcal{R}(f) = \mathbb{R}$ ' states that the range of a function is the set of real numbers; the truth of this sentence depends upon the function f being referred to.
7. SEN; CONDITIONAL. The truth of this sentence depends upon the choice of functions f and g , and the choice of x .
9. SEN; FALSE. The set $\{a\}$ is NOT an element of the set $\{a, b\}$.
11. $\mathcal{R}(f) = [0, 8]$
13. $\mathcal{R}(h) = \{1, 2, 3, 4, 5\}$

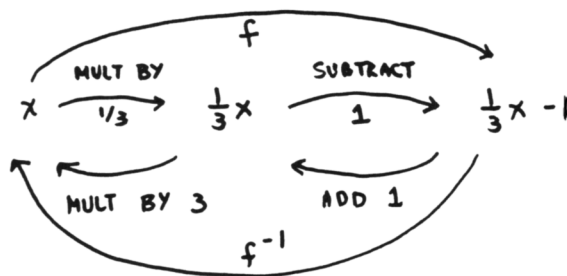
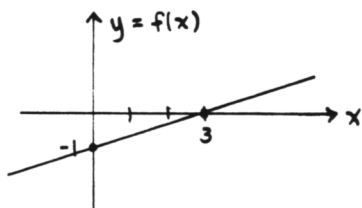
Section 2.4 One-to-One Functions and Inverse Functions

Quick Quiz:

1. The function $f(x) = x^2$ is NOT a one-to-one function. It does NOT have the property for every y , there is a unique x . That is, it does NOT pass the horizontal line test.
2. Translation: 'For every y in the range of f , there exists a unique x in the domain of f .' This is the 'one-to-one' condition for a function f .
3. One correct graph is shown below:

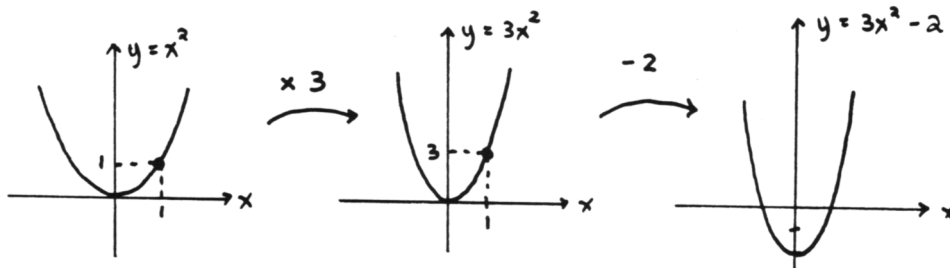


4. $f(f^{-1}(x)) = x \quad \forall x \in \mathcal{R}(f)$
 $f^{-1}(f(x)) = x \quad \forall x \in \mathcal{D}(f)$
5. The graph of f is the line shown below; it is clearly $1 - 1$. The function f takes an input x , multiplies by $\frac{1}{3}$, then subtracts 1; to 'undo' this, f^{-1} must take an input x , add 1, then divide by $\frac{1}{3}$ (that is, multiply by 3). Thus: $f^{-1}(x) = 3(x + 1)$



End-Of-Section Exercises:

1. EXP; this is a function f^{-1} , evaluated at x
3. SEN; T
5. EXP
7. SEN; T
9. EXP
11. $\mathcal{D}(f) = \mathbb{R}, \mathcal{R}(f) = (-2, \infty)$



13. $D(h) = \mathbb{R}$, $\mathcal{R}(h) = (5, \infty)$

