6.6 Integration by Parts Formula

Introduction

An attentive reader may have noticed that we have not yet learned how to integrate $\ln x$. Indeed, the integral $\int \ln x\,dx$ is a *classic* example of an integral that requires the *integration by parts* formula, which is the topic of this section. First, a derivation.

derivation of the Integration By Parts formula

The integration by parts formula is an easy consequence of the product rule for differentiation. Suppose that u and v are differentiable functions of x. Then, the product uv is also differentiable, and:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides with respect to x (and using the linearity of the integral) yields:

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

the remaining integrals absorb the constant of integration

Look at the left-hand side of this equation, and answer the following (trick) question: do we know a function whose derivative with respect to x is $\frac{d}{dx}(uv)$? Of course! The function uv has derivative $\frac{d}{dx}(uv)$! So we can replace the left-hand side by $uv + C$ to obtain:

$$uv + C = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

This equation can be simplified considerably. First, observe that the indefinite integrals remaining on the right-hand side will generate their own constant of integration, so it is not necessary to include the constant C on the left-hand side.

Furthermore, the integrals $\int u\frac{dv}{dx}\,dx$ and $\int v\frac{du}{dx}\,dx$ can be replaced by the simpler notation $\int u\,dv$ and $\int v\,du$. Thus, we have:

$$uv = \int u\,dv + \int v\,du$$

The final result is rearranged slightly, by solving for $\int u\,dv$:

Integration by Parts formula

$$\int u\,dv = uv - \int v\,du$$

This formula is commonly referred to more simply as the 'parts formula'.

EXERCISE 1 ♣ Derive the integration by parts formula, without looking at the text.

using the Integration by Parts formula;

hopefully, the new integral is easier

The idea in using the integration by parts formula is a familiar one: take a difficult integration problem, and try to *transform* it into an easier problem. When using the integration by parts formula, one takes an integral of the form $\int u\,dv$ and rewrites it in the form $uv - \int v\,du$. The hope is that the 'new' integral $\int v\,du$ is easier than the original integral $\int u\,dv$.

The general scheme is outlined below, and then illustrated in the example that follows.

a general procedure for using the parts formula

- Suppose that $\int f(x)\,dx$ cannot be solved by either elementary formulas, or substitution. It is decided to try integration by parts.
- You must *choose* u and dv to *rewrite* the integral in the form $\int u\,dv$. There will often be several possible choices for u and dv; this is the part of the problem that requires some expertise.

  A general strategy for choosing a u and dv that 'work' is presented after the example.
- From u, obtain du by differentiation.
- From dv, obtain v by integration. *Any* antiderivative can be used—usually (but not always), the constant of integration C is chosen to be zero, to obtain the simplest antiderivative.
- At this point, all the ingredients are at hand to rewrite the integral using the parts formula:

  $$\int u\,dv = uv - \int v\,du$$

  Look at the new integral $\int v\,du$. The hope is that this *new* integral $\int v\,du$ is easier to handle than the original integral $\int u\,dv$.

EXAMPLE

a classic; integrating $\int \ln x\,dx$

Problem: Find $\int \ln x\,dx$.

Solution: No previous technique seems to work here, so we are motivated to try the integration by parts formula. First, u and dv must be chosen to rewrite $\int \ln x\,dx$ in the form $\int u\,dv$.

The choices $u = \ln x$ and $dv = dx$ are made; following the example, the motivation for these choices is discussed.

Then:

CHOOSE THESE

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

COMPUTE THESE

$$\begin{aligned}\int \overbrace{\ln x}^{u}\ \overbrace{dx}^{dv} &= (\overbrace{\ln x}^{u})(\overbrace{x}^{v}) - \int \overbrace{x}^{v} \cdot \overbrace{\frac{1}{x}\,dx}^{du} \\ &= x\ln x - \int 1\,dx \\ &= x\ln x - x + C\end{aligned}$$

Check: $\dfrac{d}{dx}(x\ln x - x) = \big[x(\frac{1}{x}) + (\ln x)(1)\big] - 1 = 1 + \ln x - 1 = \ln x$

So, the result is correct.

a strategy for choosing u and dv

Here is a general strategy for choosing u and dv :

- The choice for dv must include dx. Also, since dv must be integrated to obtain v, *you must choose something for dv that you know how to integrate.* Sometimes, this consideration will completely determine the choice. (Observe that once dv is chosen, u must be everything that is left.)
- If there are several possible choices for dv, then *choose something for u that gets EASIER when you differentiate it.* This is motivated by the fact that du appears in the new integral: the simpler du is, the better.

In many problems, these two considerations will lead to a correct choice for u and dv. If not—experience, trial and error, and luck can all be factors in obtaining a correct choice for u and dv (if one exists).

return to the previous example; choosing u and dv

Reconsider the problem of finding $\int \ln x\,dx$. Here's how we arrived at the choices for u and dv :

- Choose something for dv that includes dx, and *that you know how to integrate.* We *can't* choose dv to be $\ln x\,dx$, since we don't know how to integrate this (that's the problem). So we are forced to choose $dv = dx$.
- Now, the choice for u is completely determined: u must equal everything else. Thus, $u = \ln x$.

EXAMPLE

choosing u and dv

Problem: Evaluate $\int xe^x\,dx$.

- There are several possible choices for dv here, since there are several 'pieces' that we know how to integrate. We could choose:

$$\begin{aligned} & dv = dx \\ \text{or} \quad & dv = x\,dx \\ \text{or} \quad & dv = e^x\,dx \end{aligned}$$

  Since this first consideration has not solved the 'choice' problem, we move on to the next consideration.
- Choose something for u that gets *simpler* when you differentiate it. If we choose $u = e^x$, then $\frac{du}{dx} = e^x$, which is no simpler. But if we choose $u = x$, then $\frac{du}{dx} = 1$, which is certainly simpler.
- Thus, choose $u = x$. Then dv must be everything else: $dv = e^x\,dx$. Here's how the problem is written down:

$$\begin{array}{|ll|} \hline u = x & dv = e^x dx \\ du = dx & v = e^x \\ \hline \end{array}$$

$$\begin{aligned} \int x\,e^x\,dx &= (x)(e^x) - \int e^x\,dx \\ &= x\,e^x - e^x + C \end{aligned}$$

EXERCISE 2	♣ Check that: $\frac{d}{dx}(xe^x - e^x) = xe^x$

In the following examples, use the strategy to see how we arrived at the choices for u and dv.

EXAMPLE

Problem: Evaluate $\int \frac{x}{e^x}\,dx$.

Solution:

$u = x \quad dv = e^{-x}\,dx$

$du = dx \quad v = -e^{-x}$

$$
\begin{aligned}
\int \frac{x}{e^x}\,dx &= \int xe^{-x}\,dx \\
&= -xe^{-x} - \int (-e^{-x})\,dx \\
&= -xe^{-x} + \int e^{-x}\,dx \\
&= -xe^{-x} - e^{-x} + C \\
&= -e^{-x}(x+1) + C
\end{aligned}
$$

EXERCISE 3

♣ 1. In applying the parts formula to $\int xe^{-x}\,dx$, list three possible choices for dv.

♣ 2. Corresponding to each choice for dv, what would u have to be? In which case is $\frac{du}{dx}$ simpler than u?

EXAMPLE

Problem: Evaluate $\int x^2 \ln x\,dx$.

Solution: One could choose either $dv = dx$ or $dv = x^2\,dx$, since both of these pieces can be integrated with prior techniques. If $dv = dx$ is chosen, then u must be $x^2 \ln x$, which gets much more complicated when differentiated. If $dv = x^2\,dx$ is chosen, then u must be $\ln x$, with the relatively simply derivative $\frac{1}{x}$. Thus, it is decided to initially try $dv = x^2\,dx$:

$u = \ln x \quad dv = x^2\,dx$

$du = \frac{1}{x}\,dx \quad v = \frac{x^3}{3}$

$$
\begin{aligned}
\int x^2 \ln x\,dx &= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}(\frac{1}{x})\,dx \\
&= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx \\
&= \frac{x^3}{3}\ln x - \frac{1}{3}\cdot\frac{x^3}{3} + C \\
&= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C
\end{aligned}
$$

EXERCISE 4

Use the parts formula to evaluate the following integrals. Use the 'strategy' to decide on your choices for u and dv.

♣ 1. $\int x \ln x\,dx$

♣ 2. $\int xe^{3x}\,dx$

♣ 3. $\int x^3 \ln x\,dx$

♣ 4. $\int \ln 3x\,dx$

a problem that's easier if a nonzero constant of integration is chosen when finding v

Problem: Evaluate $\int \ln(x+3)\,dx$.

Solution: We must choose $dv = dx$ and hence $u = \ln(x+3)$. If the 'traditional' approach is taken, where the constant of integration is chosen to be 0 when going from dv to v, then here's what happens:

$$\int \ln(x+3)\,dx = x\ln(x+3) - \int x \cdot \frac{1}{x+3}\,dx$$

$$u = \ln(x+3) \qquad dv = dx$$
$$du = \frac{1}{x+3}\,dx \qquad v = x$$

This is fine, except that to solve the resulting integral $\int \frac{x}{x+3}\,dx$, either a 'role-reversing' substitution or long division is required. However, if we're a bit clever, this can be avoided:

$$u = \ln(x+3) \qquad dv = dx$$
$$du = \frac{1}{x+3}\,dx \qquad v = x+3$$

$$\begin{aligned}\int \ln(x+3)\,dx &= (x+3)\ln(x+3) - \int (x+3)\frac{1}{x+3}\,dx \\ &= (x+3)\ln(x+3) - \int (1)\,dx \\ &= (x+3)\ln(x+3) - x + C\end{aligned}$$

In obtaining v, we merely need a function whose derivative with respect to x is 1 ($dv = dx \iff \frac{dv}{dx} = 1$). Usually, we use $v = x$, because it's simplest. Here, however, it was certainly to our advantage to choose a nonzero constant of integration.

EXERCISE 5

♣ 1. Check that: $\frac{d}{dx}[(x+3)\ln(x+3) - x] = \ln(x+3)$

♣ 2. Find all the antiderivatives of $3\ln(x+1)$.

♣ 3. Evaluate $\int \ln(t - \frac{1}{2})\,dt$.

EXAMPLE
repeated parts

Problem: Evaluate $\int x^2 e^x \, dx$.
Solution:

$$\boxed{\begin{array}{ll} u = x^2 & dv = e^x dx \\ du = 2x dx & v = e^x \end{array}}$$

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - \int 2x e^x \, dx \\ &= x^2 e^x - 2 \int x e^x \, dx \end{aligned}$$

$$\boxed{\begin{array}{ll} u = x & dv = e^x dx \\ du = dx & v = e^x \end{array}}$$

$$\begin{aligned} \int x e^x \, dx &= x e^x - \int e^x \, dx \\ &= x e^x - e^x + C \end{aligned}$$

Combining results:

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2(x e^x - e^x) + C \\ &= e^x (x^2 - 2x + 2) + C \end{aligned}$$

Check:

$$\begin{aligned} \frac{d}{dx}\big(e^x(x^2 - 2x + 2)\big) &= e^x(2x - 2) + e^x(x^2 - 2x + 2) \\ &= e^x(2x - 2 + x^2 - 2x + 2) \\ &= x^2 e^x \end{aligned}$$

After the first application of parts, it was noted that the resulting 'new' integral $\int x e^x \, dx$ was easier than the one started with: the power of x was knocked down by one. Thus, we were motivated to repeat the process.

It's very important to write things down neatly and carefully!

EXERCISE 6

- ♣ 1. Re-do the previous example without looking at the text.
- ♣ 2. Evaluate $\int x^2 e^{3x} \, dx$. Be sure to write a complete mathematical sentence.

EXERCISE 7

Evaluate the integral $\int \frac{x}{(1+x)^6} \, dx$ in two ways:

- ♣ 1. First, use an appropriate 'role-reversal' substitution. Differentiate to verify that you have a correct solution.
- ♣ 2. Second, use parts with $u = x$ and a corresponding dv. Differentiate to verify that you have a correct solution.
- ♣ 3. The answers obtained from the two different approaches probably look a bit different. However, they must differ by at most a constant. Express each answer as a fraction with the same denominator, so that you can better compare them.

The antidifferentiation tools studied in this chapter are summarized next for your convenience:

ANTIDIFFERENTIATION TOOLS

$F'(x) = f(x)$	F is an *antiderivative* of f
$\int f(x)\,dx$	*all* antiderivatives of f
$\int f'(x)\,dx = f(x) + C$	all antiderivatives differ by a constant
$\int \big(f(x) + g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx$	, the integral of a sum is the sum of the integrals
$\int kf(x)\,dx = k\int f(x)\,dx$	constants can be 'slid out' of the integral
$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$	Simple Power Rule for integration, $n \ne -1$
$\int \frac{1}{x}\,dx = \ln\|x\| + C$	integrating $\frac{1}{x}$
$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$	integrating e^{kx}
$\int f'(u)\frac{du}{dx}\,dx = f(u) + C$, u a function of x	substitution technique
$\int u\,dv = uv - \int v\,du$	integration by parts formula

QUICK QUIZ
sample questions

1. What is the *Integration By Parts* formula? Where does it come from?
2. Evaluate $\int \ln 2t\,dt$.
3. Evaluate $\int \ln(x-1)\,dx$.
4. What must you think of when choosing dv for use in the Parts formula?

KEYWORDS
for this section

Integration by Parts formula, derivation of the parts formula, a strategy for choosing u and dv, choosing a nonzero constant when obtaining v, repeated parts.

END-OF-SECTION EXERCISES

The purpose of these exercises is to provide you with additional practice using *all* the antidifferentiation techniques discussed thus far in this chapter. Be sure to write complete mathematical sentences. Properties of exponents and logarithms may be needed to rewrite the integrand before integrating.

1. $\displaystyle\int (e^x - 1)^2\,dx$
2. $\displaystyle\int \frac{\ln(x^2 + 2x + 1)}{x+1}\,dx$
3. $\displaystyle\int \frac{e^x}{1 + e^x}\,dx$
4. $\displaystyle\int \ln\frac{1+x}{x}\,dx$
5. $\displaystyle\int \sqrt{\frac{e^t}{2}}\,dt$
6. $\displaystyle\int \frac{x}{\sqrt{x^4 \ln x}}\,dx$