5.5 More Graphing Techniques

graphing polynomials

Since polynomials are infinitely differentiable, the only critical points and candidates for inflection points arise from places where P' and P'' are equal to zero. *If* P' and P'' can be factored, then their zeroes are easy to find; if not, the zeroes can be *approximated* using the *Intermediate Value Theorem.*

In this section, some techniques concerned with factoring polynomials are reviewed. Most of these techniques should be familiar to you from algebra, and are merely gathered here for your convenience. We begin by studying quadratic polynomials.

factorable over the integers

Let $P(x) = ax^2 + bx + c,\ \ a \neq 0$, be a quadratic polynomial. The polynomial P is '*factorable over the integers*' if

$$P(x) = (K_1x + K_2)(K_3x + K_4)\ ,$$

where the K_i are all integers.

Thus, $P(x) = 2x^2 + 5x - 3 = (2x - 1)(x + 3)$ *is* factorable over the integers, but $P(x) = x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$ *is not* factorable over the integers.

factoring $x^2 + bx + c$, b and c integers

If $P(x) = x^2 + bx + c$, where the coefficient of the x^2 term is 1, then one usually takes the approach illustrated below to try and factor P:

Problem: Factor $P(x) = x^2 + x - 6$.

Solution: A factorization of P must be of the form:

$$x^2 + x - 6 = (x + A)(x + B) = x^2 + (\overbrace{A + B}^{\text{must } = 1})x + \overbrace{AB}^{\text{must } = -6}$$

Thus, one seeks integers A and B that multiply together to give -6 (the constant term), and that add together to give 1 (the coefficient of the x term). In this case, taking $A = 3$ and $B = -2$ work, so that:

$$x^2 + x - 6 = (x + 3)(x - 2)$$

When $a \neq 1$, a similar approach can be taken, and is discussed next.

factoring $ax^2 + bx + c$, integer coefficients

Suppose that $P(x) = ax^2 + bx + c$, $a \neq 0$, has integer coefficients, and *is* factorable over the integers. That is, suppose there exist integers K_1, K_2, K_3 and K_4 for which:

$$\begin{aligned} ax^2 + bx + c &= (K_1x + K_2)(K_3x + K_4) \\ &= \overbrace{K_1K_3}^{a}\,x^2 + (\overbrace{\underbrace{K_2K_3}_{:=A} + \underbrace{K_1K_4}_{:=B}}^{b})x + \overbrace{K_2K_4}^{c} \quad \text{(multiplying out)} \end{aligned}$$

Defining $A := K_2K_3$ and $B := K_1K_4$, we see that

$$AB = (K_2K_3)(K_1K_4) = (K_1K_3)(K_2K_4) = ac$$

and:

$$A + B = K_2K_3 + K_1K_4 = b$$

What is all this saying? It says that:

Whenever a polynomial $ax^2 + bx + c$ is factorable over the integers, we can find integers A and B, where $AB = ac$ and $A + B = b$, that (we'll see) can be used to factor the polynomial for us!

The technique is illustrated in the next example.

EXAMPLE

factoring a quadratic, $a \neq 1$

Problem: Factor $8x^2 - 10x - 3$.

Solution: We seek integers A and B satisfying

$$AB = (\text{coefficient of } x^2 \text{ term}) \cdot (\text{constant term})$$

and:

$$A + B = \text{coefficient of } x \text{ term}$$

Thus, we want:

$$AB = (8)(-3) = -24 \quad \text{and} \quad A + B = -10$$

Choosing $A = -12$ and $B = 2$ works. Then:

$$\begin{aligned} 8x^2 - 10x - 3 &= 8x^2 + (2x - 12x) - 3 && \text{(rewrite middle term)} \\ &= (8x^2 + 2x) + (-12x - 3) && \text{(regroup)} \\ &= 2x(4x + 1) - 3(4x + 1) && \text{(factor each group)} \\ &= (2x - 3)(4x + 1) && \text{(factor out } (4x + 1)) \end{aligned}$$

Note that when the middle term is rewritten as a sum, the order *does not matter*:

$$\begin{aligned} 8x^2 - 10x - 3 &= 8x^2 + (-12x + 2x) - 3 && \text{(rewrite middle term)} \\ &= (8x^2 - 12x) + (2x - 3) && \text{(regroup)} \\ &= 4x(2x - 3) + (2x - 3) && \text{(factor each group)} \\ &= (4x + 1)(2x - 3) && \text{(factor out } (2x - 3)) \end{aligned}$$

EXERCISE 1

Use the technique described above to factor the following quadratics.

♣ 1. $3x^2 + 2x - 1$

♣ 2. $10x^2 - 13x - 3$

♣ 3. $14x^2 + 19x - 3$

★

When is $ax^2 + bx + c$, with integer coefficients, factorable over the integers?

Here's a precise statement of the factoring result discussed above:

THEOREM. Let $P(x) = ax^2 + bx + c$ have integer coefficients, $a \neq 0$. Then, P is factorable over the integers if and only if there exist integers A and B with $AB = ac$ and $A + B = b$.

Idea of Proof. It has already been shown that if P is factorable over the integers, then integers A and B with the desired property exist.

The other direction uses the fact that a polynomial with integer coefficients is factorable over $\mathbb{Z}$ iff it is factorable over $\mathbb{Q}$ (see, e.g., John B. Fraleigh, A First Course in Abstract Algebra, third edition, page 280). Suppose integers A and B exist with $AB = ac$ and $A+B = b$. If $c = 0$, then $ax^2+bx = x(ax+b)$ is factorable over $\mathbb{Z}$. Suppose $c \neq 0$. Then, since $a \neq 0$, and $AB = ac$, both A and B are nonzero. Further, $AB = ac \implies \frac{A}{a} = \frac{c}{B}$. Then:

$$\begin{aligned} ax^2 + bx + c &= ax^2 + (A + B)x + c \\ &= (ax^2 + Ax) + (Bx + c) \\ &= ax(x + \frac{A}{a}) + B(x + \frac{c}{B}) \\ &= (ax + B)(x + \frac{c}{B}) \end{aligned}$$

Thus, P is factorable over $\mathbb{Q}$, and hence over $\mathbb{Z}$. ∎

a technique that always works;

using the quadratic formula

The *quadratic formula* can always be used to factor *any quadratic polynomial*, whether or not it is factorable over the integers. Recall that the *quadratic formula* says that the equation $ax^2 + bx + c = 0$, $a \neq 0$, has solutions x_1 and x_2 given by:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The '+' sign gives one solution; the '−' sign gives the second solution.

These zeroes provide the factors of the polynomial:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

Note that you must supply the constant factor a yourself.

EXAMPLE

factoring a quadratic by using the quadratic formula

Problem: Factor $8x^2 + 5x - 3$, using the quadratic formula.

Solution: First, find the roots of this quadratic. That is, solve:

$$8x^2 + 5x - 3 = 0$$

By the quadratic formula:

$$\begin{aligned} x_{1,2} &= \frac{-5 \pm \sqrt{5^2 - 4(8)(-3)}}{2(8)} \\ &= -1\ ,\ \frac{3}{8} \end{aligned}$$

Since -1 is a root, $x-(-1)=x+1$ is a factor.
Since $\frac{3}{8}$ is a root, $x-\frac{3}{8}$ is a factor.
Only the constant factor need be supplied:

$$\begin{aligned} 8x^2+5x-3 &= 8(x+1)(x-\frac{3}{8}) \\ &= (x+1)8(x-\frac{3}{8}) \\ &= (x+1)(8x-3) \end{aligned}$$

♣ Use the technique discussed earlier to factor $8x^2+5x-3$.

EXERCISE 2 ♣ Use the quadratic formula to factor each polynomial from Exercise 1.

EXAMPLE

graphing a more complicated polynomial

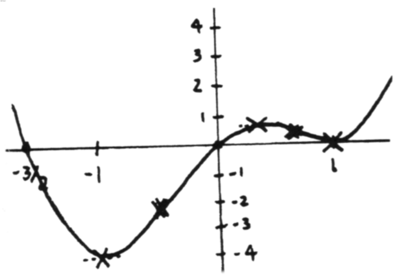
Problem: Completely graph $f(x)=(x-1)^2(2x+3)x$.

- Plot a few points:

x	P(x)
1	0
-3/2	0
0	0
2	14
-1	-4

x	P(x)
3/8	≃ .55
.72	≃ .25
-.47	≃ -2.09

- Find the first derivative. Use the 'generalized product rule': $\frac{d}{dx}(ABC)=A'BC+AB'C+ABC'$



$$\begin{aligned} f'(x) &= 2(x-1)(2x+3)x+(x-1)^2(2)x+(x-1)^2(2x+3)(1) \\ &= (x-1)\big[2x(2x+3)+2x(x-1)+(x-1)(2x+3)\big] \\ &= (x-1)(8x^2+5x-3) \\ &= (x-1)(x+1)(8x-3) \end{aligned}$$

Thus, $f'(x)=0$ when $x=1,\,-1,\,\frac{3}{8}$. Find the corresponding function values, and add these points to the table of points started above. Plot the points with a $\times$.

- Find the second derivative:

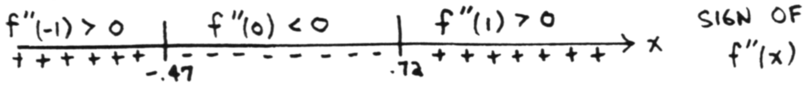
$$\begin{aligned} f''(x) &= (1)(x+1)(8x-3)+(x-1)(1)(8x-3)+(x-1)(x+1)(8) \\ &= 24x^2-6x-8 \\ &= 2(12x^2-3x-4) \end{aligned}$$

Using the quadratic formula, the solutions of $12x^2-3x-4=0$ are:

$$x_1=\frac{3+\sqrt{201}}{24}\approx 0.72 \quad \text{and} \quad x_2=\frac{3-\sqrt{201}}{24}\approx -0.47$$

Find the corresponding function values, and plot these points with a ⋇.

- Sign of f'':



  Use this concavity information to fill in the graph.
- Behavior at infinity: As $x \to \pm\infty$, $f(x) \approx 2x^4 \to \infty$, which agrees with the graph.

some final results

The remainder of this section is a collection of useful results and techniques concerning polynomials. These may be familiar to you from algebra. They are merely gathered here for your convenience.

RATIONAL ROOT THEOREM

Let $P(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial with *integer* coefficients. Suppose that $a_n \neq 0$ and $a_0 \neq 0$.

If P has a rational zero $\frac{p}{q}$ (in lowest terms), then p is a factor of a_0 and q is a factor of a_n.

What if $a_0 = 0$?

Observe that if $a_0 = 0$ and $a_1 \neq 0$, then:

$$P(x) = x(\overbrace{a_n x^{n-1} + \cdots + a_2 x + a_1}^{\tilde{P}(x)})$$

Apply the Rational Root Theorem to $\tilde{P}(x)$.

★ **PROOF** *of the Rational Root Theorem*

Proof. The notation $a|b$ (read as 'a divides b') means that a is a factor of b. Suppose $\frac{p}{q}$ is a rational root in lowest terms, so:

$$a_n(\frac{p}{q})^n + a_{n-1}(\frac{p}{q})^{n-1} + \cdots + a_1(\frac{p}{q}) + a_0 = 0$$

Multiplication by q^n yields:

$$a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0 \qquad (*)$$

Observe that all terms except the last have a factor of p. Then:

$$p(a_n p^{n-1} + \cdots + a_1 q^{n-1}) = -a_0 q^n$$

Since p divides the left-hand side, it must divide the right-hand side. But $p \not| q$, so $p \not| q^n$, so it must be that $p|a_0$.

For the remaining result, observe that every term in (*) except the first has a factor of q. Repeat the argument, with obvious changes. ∎

negating 'A and B'

The Rational Root Theorem is an implication (with some additional hypotheses):

IF P has a rational zero $\frac{p}{q}$ (in lowest terms),
THEN (p is a factor of a_0) and (q is a factor of a_n).

The conclusion of this implication is a sentence of the form 'A and B'. Thus, to find the contrapositive of this implication, one must negate 'A and B'. How is this done?

Use your intuition: 'A and B' is true only when *both* A and B are true. So when is 'A and B' false? When A is false, or B is false. Precisely,

$$\text{not}(A \text{ and } B) \iff (\text{not}\, A) \text{ or } (\text{not}\, B) \ ,$$

as the truth table below confirms:

A	B	A and B	not (A and B)	not A	not B	(not A) or (not B)
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

logical symbols:
∧ for 'and'
∨ for 'or'
¬ for 'not'
DeMorgan's laws

The sentence 'A and B' can be written as $A \wedge B$. The symbol $\wedge$ is a synonym for the mathematical word 'and'.

The sentence 'A or B' can be written as $A \vee B$. The symbol $\vee$ is a synonym for the mathematical word 'or'.

The sentence 'not A' can be written as $\neg A$. The symbol $\neg$ is a synonym for the mathematical word 'not'.

With this notation, the previous logical equivalence can be more simply written as:

$$\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$$

In the next exercise, you are asked to prove that:

$$\neg(A \vee B) \iff (\neg A) \wedge (\neg B)$$

These two logical equivalences are commonly known as *DeMorgan's Laws.*

EXERCISE 3

♣ Prove that:

$$\neg(A \vee B) \iff (\neg A) \wedge (\neg B)$$

That is, make a truth table which shows that $\neg(A \vee B)$ and $(\neg A) \wedge (\neg B)$ always have the same truth values.

Now, the contrapositive of the sentence:

IF P has a rational zero $\frac{p}{q}$ (in lowest terms),
THEN (p is a factor of a_0) and (q is a factor of a_n)

is:

IF (p is not a factor of a_0) or (q is not a factor of a_n),
THEN $\frac{p}{q}$ is not a zero of P

This latter sentence tells us that the only *candidates* for rational roots of P are numbers of the form $\frac{p}{q}$, where p is a factor of the constant term, and q is a factor of the leading coefficient. The next example illustrates how this information is used.

EXAMPLE
using the Rational Root Theorem

Problem: Find all rational roots of $P(x) = 14x^4 - x^3 - 17x^2 + x + 3$. Use these roots to factor P as completely as possible.

Solution: The leading coefficient is 14, with factors: ± 1, ± 2, ± 7, ± 14

The constant term is 3, with factors: ± 1 and ± 3

Thus, if $\frac{p}{q}$ is a root of P, it must be that:

$$p \in \{\pm 1,\, \pm 3\} \quad \text{and} \quad q \in \{\pm 1,\, \pm 2,\, \pm 7,\, \pm 14\}$$

That is:

$$\frac{p}{q} \in \{\pm 1,\, \pm\frac{1}{2},\, \pm\frac{1}{7},\, \pm\frac{1}{14},\, \pm 3,\, \pm\frac{3}{2},\, \pm\frac{3}{7},\, \pm\frac{3}{14}\}$$

Each candidate is checked:

$$\begin{aligned}
P(1) &= 14(1)^4 - 1^3 - 17(1)^2 + 1 + 3 = 0 && \text{(root 1, factor } x-1\text{)}\\
P(-1) &= 14(-1)^4 - (-1)^3 - 17(-1)^2 + (-1) + 3 = 0 && \text{(root } -1\text{, factor } x+1\text{)}\\
P(\frac{1}{2}) &= \cdots = 0 && \text{(root } \tfrac{1}{2}\text{, factor } x-\tfrac{1}{2}\text{)}\\
P(-\frac{1}{2}) &= \cdots \ne 0 && (-\tfrac{1}{2} \text{ is not a root})\\
&\vdots
\end{aligned}$$

Continuing, it is found that $P(1) = P(-1) = P(-\frac{3}{7}) = P(\frac{1}{2}) = 0$. This information is used to factor P:

$$\begin{aligned}
P(x) &= 14(x-\frac{1}{2})(x+\frac{3}{7})(x-1)(x+1)\\
&= 2(x-\frac{1}{2})7(x+\frac{3}{7})(x-1)(x+1)\\
&= (2x-1)(7x+3)(x-1)(x+1)
\end{aligned}$$

Note that we had to supply the constant factor of 14 ourselves.

EXAMPLE
using the Rational Root Theorem

Problem: Find all rational roots of $P(x) = x^4 - 2x^2 - 3x - 2$. Use these roots to factor P as completely as possible.

Solution: If $\frac{p}{q}$ is a rational root, then:

$$p \in \{\pm 1,\, \pm 2\} \quad \text{and} \quad q \in \{\pm 1\}$$

Thus:

$$\frac{p}{q} \in \{\pm 1,\, \pm 2\}$$

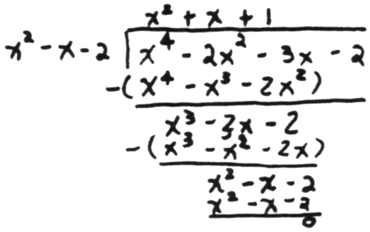
Indeed:

$$\begin{aligned}
P(1) &= 1 - 2 - 3 - 2 \ne 0\\
P(-1) &= 1 - 2 + 3 - 2 = 0\\
P(2) &= 16 - 8 - 6 - 2 = 0\\
P(-2) &= 16 - 8 + 6 - 2 \ne 0
\end{aligned}$$

Thus, -1 and 2 are roots, so:

$$P(x) = (x+1)(x-2)(????) = (x^2 - x - 2)(????)$$

Use long division to find the remaining factor:



Thus:

$$P(x) = (x+1)(x-2)(x^2 + x + 1)$$

An application of the quadratic formula shows that the roots of $x^2 + x + 1$ are not real numbers. Thus, P cannot be factored any further, using only real numbers.

EXERCISE 4

♣ 1. Refer to the previous example. Find two more polynomials, different from P, that have precisely the same candidates for rational roots. (Hint: Only the leading coefficient and the constant term are used to find the candidates.)

♣ Use the rational root theorem to find all rational roots of the following polynomials. Use this information to factor the polynomial as completely as possible.

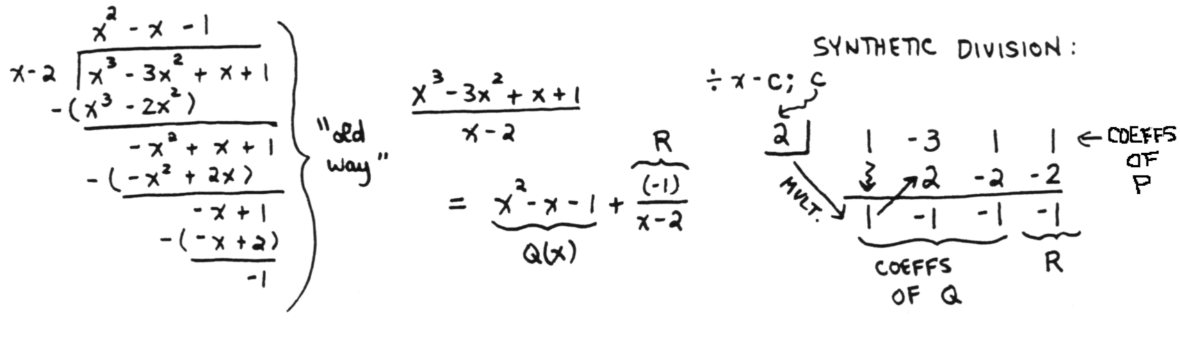
♣ 2. $5x^3 - 3x^2 - 12x - 4$

♣ 3. $4x^4 + 5x^3 - 2x^2 + 5x - 6$

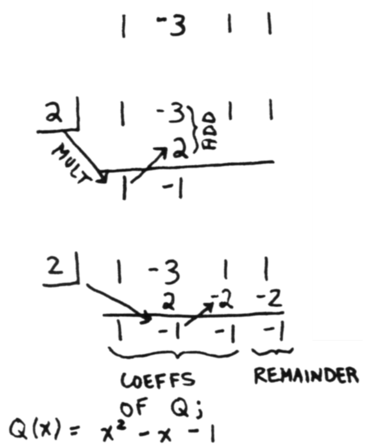
♣ 4. $3x^4 - x^3 + 12x^2 - 4x$ (Hint: First factor out an x. Then, apply the Rational Root Theorem to the remaining polynomial.)

SYNTHETIC DIVISION

Finding $\frac{P(x)}{x-c}$ via long division involves a lot of redundancy. Synthetic division suppresses all this redundancy and results in a useful tool for factoring. The process is illustrated below.



Here's how synthetic division is used to compute $\dfrac{P(x)}{x-c}$:



- Make sure P is written with decreasing powers of x.
- Write down the coefficients of P. Be sure to include 0 for any missing terms.
- To divide by $x-c$, put the number 'c' in a box to the left of the coefficients.

  For example, to divide by $x-2$, put a '2' in the box. To divide by $x+3 = x-(-3)$, put a '-3' in the box.
- Bring down the first coefficient.
- Multiply by c, and add to the next coefficient of P, as shown.
- Repeat as necessary. You have now computed

$$\frac{P(x)}{x-c} = Q(x) + \frac{R}{x-c} \iff P(x) = (x-c)Q(x) + R\ ;$$

you need only read off the coefficients of Q and the remainder R.

The last number computed is the remainder R. The preceding numbers are the coefficients of Q. Observe that the degree of Q is always one less than the degree of R.

REMAINDER THEOREM	If P is a polynomial and $P(x) = (x-r)Q(x) + R$, then $P(r) = R$.

The proof is trivial! $P(r) = (r-r)Q(r) + R = 0 \cdot Q(r) + R = R$. ∎

Usually, to evaluate a polynomial at a number r, we substitute r into the formula for P and crunch away. This theorem gives an alternate approach! It says that, to evaluate P at r, one can instead divide $P(x)$ by $x-r$; the remainder is precisely $P(r)$.

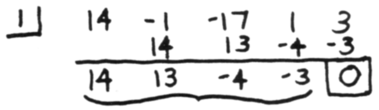
The Remainder Theorem, together with synthetic division, gives an efficient way to evaluate polynomials, as illustrated next.

EXAMPLE

using synthetic division and the Remainder Theorem to evaluate polynomials

Problem: Evaluate $P(x) = 14x^4 - x^3 - 17x^2 + x + 3$ at $x = 1$ and $x = -2$.

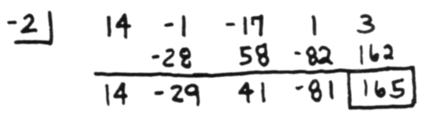
Solution: To find $P(1)$, use synthetic division to divide by $x-1$:



The remainder is 0, so $P(1) = 0$. Checking:

$$P(1) = 14 - 1 - 17 + 1 + 3 = 0$$

To find $P(-2)$, use synthetic division to divide by $x+2$:



The remainder is 165. Thus, $P(-2) = 165$. This was considerably easier than computing:

$$P(-2) = 14(-2)^4 - (-2)^3 - 17(-2)^2 + (-2) + 3$$

EXERCISE 5

Use synthetic division and the Remainder Theorem to evaluate the following polynomial at the specified values of x.

♣ $P(x) = x^4 - 2x^2 - 3x - 2$; $x = 1, -1, 2, -2$

Two additional tools for gaining information about the zeroes of polynomials are *Descartes' Rule of Signs* and the *Upper and Lower Bound Theorem*. Check your algebra book for more information.

QUICK QUIZ
sample questions

1. Factor $3x^2-2x-8$, by first finding numbers A and B that satisfy $AB = ???$ and $A + B = ???$
2. Factor $3x^2 - 2x - 8$, by using the Quadratic Formula.
3. What are the candidates for the rational roots of $P(x) = x^7 - 2x^5 + 2$?
4. Negate: A and B
5. Use the Remainder Theorem to find $P(1)$ if $P(x) = x^5 - 3x^2 + 2x - 1$.

KEYWORDS
for this section

Factorable over the integers, techniques for factoring $ax^2 + bx + c$, using the quadratic formula to factor ax^2+bx+c, the Rational Root Theorem, the symbols $\wedge$, $\vee$, $\neg$, negating $A \wedge B$ and $A \vee B$, DeMorgan's Laws, synthetic division, the Remainder Theorem.

END-OF-SECTION EXERCISES

♣ Use all available techniques to factor the following polynomials as completely as possible over $\mathbb{R}$.

1. $P(x) = 2x^3 - 3x^2 - 3x - 5$
2. $P(x) = 2x^6 - 4x^5 + 3x^4 - 2x^3 + x^2$
3. $P(x) = x^4 - 5x^2 + 6$
4. $P(x) = x^3 + x^2 - x$