4.3 Some Very Basic Differentiation Formulas

Introduction

If a differentiable function f is quite simple, then it *is* possible to find f' by using the definition of derivative directly:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

However, this process is quite tedious. Also, as f gets more complicated, the limit gets increasingly more difficult to evaluate.

In this section, some differentiation formulas are developed to make life easier.

First, some notation:

NOTATION for the derivative

prime notation;

f has derivative f'

There are several notations used for the derivative.

So far, the *prime* notation has been used: if f is differentiable at x, then the slope of the tangent line at the point $(x, f(x))$ is the number $f'(x)$. The *name* of the derivative function is f'; $f'(x)$ is the function f', evaluated at x.

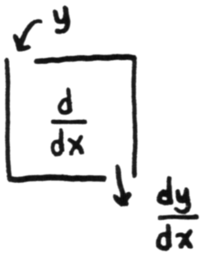
If y is a differentiable function of x, then its derivative can be denoted, using prime notation, by y'. For example, if $y = x^2$, then $y' = 2x$. If it is desired to emphasize that y' is being evaluated at a particular input c, one can write $y'(c)$.

NOTATION for the derivative

Leibniz notation;

y has derivative $\frac{dy}{dx}$;

$\frac{dy}{dx}$ evaluated at c is denoted by either $\frac{dy}{dx}(c)$ or $\frac{dy}{dx}|_{x=c}$



If y is a differentiable function of x, then its derivative can alternately be denoted by $\frac{dy}{dx}$. This is the *Leibniz* notation for the derivative. Read '$\frac{dy}{dx}$' as '*dee y, dee x*'.

For example, if $y = x^2$, then $\frac{dy}{dx} = 2x$. Again, if it is desired to emphasize that $\frac{dy}{dx}$ is being evaluated at a particular input c, one can write $\frac{dy}{dx}(c)$ or $\frac{dy}{dx}|_{x=c}$. These latter two expressions can both be read as: '*dee y, dee x, evaluated at c*'. In particular, the vertical bar '|' is read as '*evaluated at*'.

Similarly, if f is a differentiable function of x, its derivative in Leibniz notation is $\frac{df}{dx}$ (read as '*dee f, dee x*'). If one wants to emphasize that this derivative is being evaluated at a particular value, say c, then one writes $\frac{df}{dx}(c)$ or $\frac{df}{dx}|_{x=c}$.

One problem with Leibniz notation is that the *name of the function* and an *output of the function* are confused. When one says:

if $y = x^2$, then $\frac{dy}{dx} = 2x$,

the symbol $\frac{dy}{dx}$ is really being used as *both* the function name and its output. Strictly speaking, one should write: if $y = x^2$, then $\frac{dy}{dx}(x) = 2x$. However, this is not common practice.

an important use of Leibniz notation: the operator $\frac{d}{dx}$

The notation $\frac{d}{dx}$ can be used to denote an instruction: $\frac{d}{dx}$ acts on a differentiable function of x to produce its derivative.

For example, one can write:

$$\frac{d}{dx}(3x-1) = 3 \quad \text{and} \quad \frac{d}{dt}(t^2) = 2t \quad \text{and} \quad \frac{d}{dz}(2z+1) = 2$$

This '$\frac{d}{dx}$' notation is often used in stating differentation formulas. Also, it is convenient if you are asked to differentiate a function that is not given a name.

EXERCISE 1

practice with notation

Let $f(x) = x^2$.

Rewrite the following sentences about f, using prime notation.

♣ 1. $\frac{df}{dx} = 2x$

♣ 2. $\frac{df}{dx}(3) = 6$

♣ 3. $\frac{df}{dx}|_{x=3} = 6$

♣ 4. $\frac{df}{dt} = 2t$

♣ 5. $\frac{df}{dt}(3) = 6$

♣ 6. $\frac{df}{dt}|_{t=3} = 6$

Rewrite the following sentences using Leibnitz notation.

♣ 7. $f'(x) = 2x$

♣ 8. $f'(3) = 6$

♣ 9. $f'(t) = 2t$

compiling some differentiation tools

We now begin to compile some tools that will help us differentiate functions more easily.

DIFFERENTIATION TOOL

the derivative of a constant is 0

Let $f(x) = k$, for $k \in \mathbb{R}$. Then $f'(x) = 0$.

alternate notation

This rule can be rewritten, using the '$\frac{d}{dx}$' operator, as follows:
For every real number k:

$$\frac{d}{dx}(k) = 0$$

PROOF

Proof. Let $f(x) = k$, for $k \in \mathbb{R}$. Then, for every x:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{k - k}{h} = 0$$

Thus, $f'(x) = 0$. ▮

EXAMPLE

Remember that the symbol '▮' merely marks the end of the proof.
If $f(x) = \sqrt{\pi^2 - 5}$, then $f'(x) = 0$.
If $y = e - 3$, then $\frac{dy}{dx} = 0$.
$\frac{d}{dx}\left(\frac{\sqrt{7}}{3\sqrt{2}}\right) = 0$
If $f(x) = a + b$, where a and b are constants, then $f'(x) = 0$.

EXERCISE 2

Rewrite each of these examples, using alternate notation.

DIFFERENTIATION TOOL

constants can be 'slid' out of the differentiation process

Suppose that f is differentiable at x, and let $k \in \mathbb{R}$. Recall that the function kf is defined by the rule:

$$(kf)(x) := k \cdot f(x)$$

Then:

$$(kf)'(x) = k \cdot f'(x)$$

In words, *the derivative of a constant times a differentiable function is the constant, times the derivative of the function.*

alternate notation

This rule can be rewritten, using a mixture of the '$\frac{d}{dx}$' operator and prime notation, as:

$$\frac{d}{dx}\big(kf(x)\big) = k \cdot f'(x)$$

PROOF

Proof. Let f be differentiable at x, and let $k \in \mathbb{R}$. It is necessary to show that the function given by $(kf)(x) = k \cdot f(x)$ is differentiable at x.

$$\begin{aligned}
&\lim_{h\to 0} \frac{(kf)(x+h) - (kf)(x)}{h} \\
&\quad = \lim_{h\to 0} \frac{kf(x+h) - kf(x)}{h} && \text{(defn of } kf) \\
&\quad = \lim_{h\to 0} k \cdot \frac{f(x+h) - f(x)}{h} && \text{(factor out } k) \\
&\quad = k \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && \text{(prop. of limits, } f \text{ diff. at } x) \\
&\quad = k \cdot f'(x) && (f \text{ is diff at } x)
\end{aligned}$$

Thus, the function kf is differentiable at x, and has derivative given by:

$$(kf)'(x) = k \cdot f'(x) \qquad \blacksquare$$

What made this proof work? Properties of limits!

Observe what made this proof work: since we knew, a priori, that f was differentiable at x (so that $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists), we were able to use the property of *limits* to slide the constant out. The properties of limits will play a crucial role in the proofs of *all* the differentiation formulas.

EXAMPLE

If $f(x) = 2x$, then $f'(x) = 2 \cdot \frac{d}{dx}(x) = 2(1) = 2$.
If h is differentiable at x, and $f(x) = \sqrt{2}h(x)$, then $f'(x) = \sqrt{2}h'(x)$.
If $y = \frac{1}{2t} = \frac{1}{2} \cdot \frac{1}{t}$, then $\frac{dy}{dt} = \frac{1}{2} \cdot \frac{d}{dt}\big(\frac{1}{t}\big)$. (This last example can be completed after the statement of another differentiation tool, the *Simple Power Rule*.)

DIFFERENTIATION TOOL

differentiating sums and differences

Suppose that both f and g are differentiable at x. Then the functions $f + g$ and $f - g$ are also differentiable at x, and:

$$(f+g)'(x) = f'(x) + g'(x)$$
$$(f-g)'(x) = f'(x) - g'(x)$$

In words, *the derivative of a sum is the sum of the derivatives*, and *the derivative of a difference is the difference of the derivatives.*

alternate notation

This rule can be rewritten, using a mixture of the '$\frac{d}{dx}$' operator and prime notation, as:

$$\frac{d}{dx}\big(f(x)+g(x)\big) = f'(x)+g'(x)$$

PROOF

Proof. It is shown first that, under the stated hypotheses, $f+g$ is differentiable at x.

Recall that the function $f+g$ is defined by the rule $(f+g)(x) := f(x)+g(x)$.

Since, by hypothesis, both f and g are differentiable at x, it is known that $f'(x)$ and $g'(x)$ exist.

Then:

$$\begin{aligned}
&(f+g)'(x) \\
&\quad := \lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h} && \text{(defn. of derivative)}\\
&\quad = \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} && \text{(defn of } f+g\text{)}\\
&\quad = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h} && \text{(regroup)}\\
&\quad = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} && \text{(limit of a sum, hypotheses)}\\
&\quad = f'(x)+g'(x)
\end{aligned}$$

To see that $f-g$ is differentiable at x, we can now cite earlier results. Note that:

$$(f-g)(x) := f(x)-g(x) = f(x)+(-g(x)) = f(x)+(-g)(x)$$

So, the function $f-g$ can be written as a sum of two functions, with names f and $-g$. Then:

$$\begin{aligned}
(f-g)'(x) &= f'(x)+(-g)'(x) && \text{(Why?)}\\
&= f'(x)+(-g'(x)) && \text{(Why?)}\\
&= f'(x)-g'(x) \quad \blacksquare
\end{aligned}$$

EXERCISE 3

♣ 1. Prove the previous result yourself, without looking at the book. You could be asked to write down a precise proof on an in-class exam.

♣ 2. Under what hypotheses is the limit of a sum equal to the sum of the limits? Was this result used in the previous proof? Where? Were the hypotheses met?

♣ 3. Re-prove the fact that $(f-g)'(x) = f'(x)-g'(x)$ (under suitable hypotheses), but this time DON'T cite earlier results. Just use the definition of derivative.

Does the rule apply when there are more than 2 terms?

Although the previous result is stated for only 2 terms, does it tell us that, say,

$$(f+g+h)'(x) = f'(x) + g'(x) + h'(x) \ ,$$

providing that f, g and h are all differentiable at x? Of course! Just pull out the old '*treat it as a singleton*' trick:

$$\begin{aligned}(f+g+h)'(x) &= \big((f+g)+h\big)'(x) && \text{(group)}\\ &= (f+g)'(x) + h'(x) && \text{(use result once)}\\ &= f'(x)+g'(x)+h'(x) && \text{(use result again)}\end{aligned}$$

EXERCISE 4

♣ Prove that, under suitable hypotheses:

$$(f+g+h+k)'(x) = f'(x) + g'(x) + h'(x) + k'(x)$$

SIMPLE POWER RULE

differentiating x^n

For all positive integers n:

$$\frac{d}{dx}x^n = nx^{n-1}$$

More generally, if n is a real number, and I is any interval on which *both* x^n and nx^{n-1} are defined, then x^n is differentiable on the interval I, and:

$$\frac{d}{dx}x^n = nx^{n-1}$$

EXAMPLE

Here are some very basic applications of the Simple Power Rule:

- If $f(x) = x^2$, then $f'(x) = 2x^{2-1} = 2x$. Here, the Simple Power Rule was applied with $n = 2$.
- $\frac{d}{dx}x^3 = 3x^{3-1} = 3x^2$
- If $y = x^{1007}$, then $\frac{dy}{dx} = 1007x^{1006}$. Here, the Simple Power Rule was applied with $n = 1007$.
- The slope of the tangent line to the graph of $f(x) = x^7$ at the point $(2, 2^7)$ is $f'(2) = 7(2^6)$.

EXAMPLE

rewriting the function, to make it 'fit' the Simple Power Rule

Here are some more advanced applications of the Simple Power Rule. The Simple Power Rule is used whenever the function being differentiated looks like (or *can be made to look like*) x^n. The laws of exponents, and fractional exponent notation, are used extensively to rewrite functions, to get them into a form where the Simple Power Rule can be applied. The Algebra Review in this section reviews the necessary tools.

Problem: Differentiate $f(x) = \frac{1}{x}$.

Solution: Rewrite the function as $f(x) = x^{-1}$. Taking $n = -1$ in the Simple Power Rule, one obtains:

$$f'(x) = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

On what interval(s) is this formula valid? It is valid on any interval for which BOTH $\frac{1}{x}$ and $-\frac{1}{x^2}$ are defined. Both expressions are defined on $\mathbb{R} - \{0\}$. Thus, the formula is valid for all real numbers, except 0.

EXAMPLE

Problem: Differentiate $y = \sqrt{x}$.

Solution: Rewrite y, using fractional exponents, as $y = x^{1/2}$. Taking $n = \frac{1}{2}$ in the Simple Power Rule, one obtains:

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2} \cdot \frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$$

On what interval(s) is this formula valid? The expression $\sqrt{x}$ is defined for $x \ge 0$. The expression $\frac{1}{2\sqrt{x}}$ is defined for $x > 0$. BOTH expressions are defined on $(0, \infty)$. Thus, the formula is valid for all positive real numbers.

put the derivative in a form that matches the original function form

It is always a good idea to put the derivative in a form that agrees, as closely as possible, with the form of the original function. Since the original function in this example was given in radical form, $y = \sqrt{x}$, the derivative was also rewritten in radical form, $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.

$\frac{d}{dx}kx^n = nkx^{n-1}$

Using both the Simple Power Rule and the fact that constants can be 'slid out' of the differentiation process yields an extremely useful formula:

$$\frac{d}{dx}kx^n = k\frac{d}{dx}x^n = k(nx^{n-1}) = knx^{n-1}$$

Thus, for example:

- If $f(x) = 3x^2$, then $f'(x) = 6x$.
- $\frac{d}{dx}\pi x^{11} = 11\pi x^{10}$
- If $y = \sqrt{2}x$, then $\frac{dy}{dx} = (1)(\sqrt{2})x^{1-1} = \sqrt{2}x^0 = \sqrt{2}$.

  It is not necessary to write out all these steps. You should be able to recognize $y = kx$ as a *line* that has slope k. Thus, $\frac{dy}{dx} = k$.
- The slope of the tangent line to the graph of $y = 3x^5$ at the point $(1, 3)$ is $\frac{dy}{dx}|_{x=1}$. Here, $\frac{dy}{dx} = 15x^4$, so that $\frac{dy}{dx}|_{x=1} = 15(1)^4 = 15$.

EXERCISE 5

practice with the Simple Power Rule

For each of the functions listed below, do the following:

- Write the function in the form $f(x) = x^n$.
- Differentiate, using the Simple Power Rule.
- On what interval(s) is the formula obtained for the derivative valid?
- Find the equation of the tangent line to the graph of f when $x = 1$.

♣ 1. $f(x) = \sqrt[3]{x}$

♣ 2. $f(x) = \frac{1}{\sqrt{x}}$

♣ 3. $f(x) = \sqrt{x}\sqrt[3]{x^2}$

idea of proof of the Simple Power Rule

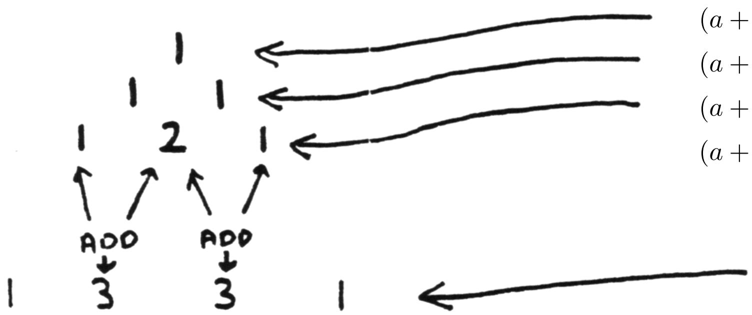
When the exponent is a positive integer, the idea of the proof of the Simple Power Rule is very simple. This idea is illustrated by considering a special case: Show that if $f(x) = x^3$, then $f'(x) = 3x^2$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{(x^3 + \overbrace{3x^2h}^{\text{one factor of } h} + \overbrace{3xh^2 + h^3}^{\text{more than one } h}) - x^3}{h} \qquad (\text{expand } (x+h)^3) \\
&= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2) \\
&= 3x^2
\end{aligned}
$$

A brief review of Pascal's Triangle, a tool for easily expanding $(a+b)^n$ for positive integers n, will enable you to repeat this argument for higher values of n.

Pascal's Triangle

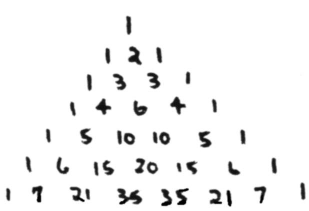
Let a and b be any real numbers. Observe the following pattern:



$$
\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= (1)a + (1)b \\
(a+b)^2 &= (1)a^2 + 2ab + (1)b^2 \\
(a+b)^3 &= (a+b)(a+b)^2 \\
&= (a+b)(a^2 + 2ab + b^2) \\
&= a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3 \\
&= (1)a^3 + 3a^2b + 3ab^2 + (1)b^3
\end{aligned}
$$

A 'triangle' is formed. Each new row is easily obtained from the previous row by simple addition. It can be proven (★ say, by induction) that this pattern continues forever.

finding $(x+h)^7$



For example, suppose we want to expand $(x+h)^7$. Long multiplication would be extremely tedious. Instead, first write the appropriate *types* of terms in the expansion. Each term has variable part x^ih^j, where the exponents add up to 7. The first term has x^7 and h^0; the second x^6 and h^1, the third term has x^5 and h^2, and so on. So we get the term types:

$$x^7 \quad x^6h \quad x^5h^2 \quad x^4h^3 \quad x^3h^4 \quad x^2h^5 \quad xh^6 \quad h^7$$

Now, get the correct coefficients from Pascal's triangle (from the row beginning with the numbers '1 7 ...'):

$$(1)x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3 + 35x^3h^4 + 21x^2h^5 + 7xh^6 + (1)h^7$$

Thus:

$$(x+h)^7 = x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3 + 35x^3h^4 + 21x^2h^5 + 7xh^6 + h^7$$

EXERCISE 6

♣ 1. Use Pascal's triangle to expand $(x+h)^9$.

♣ 2. Use Pascal's triangle to expand $(x-h)^4$.
Hint: Write $(x-h)^4 = (x+(-h))^4$, so the appropriate term 'types' are:

$$x^4 \quad x^3(-h) \quad x^2(-h)^2 \quad x(-h)^3 \quad (-h)^4$$

♣ 3. Prove that if $f(x) = x^4$, then $f'(x) = 4x^3$.

★★

The complete proof of the Simple Power Rule would take several pages, and we do not yet have at our disposal all the necessary tools. However, a sketch of the proof is as follows:

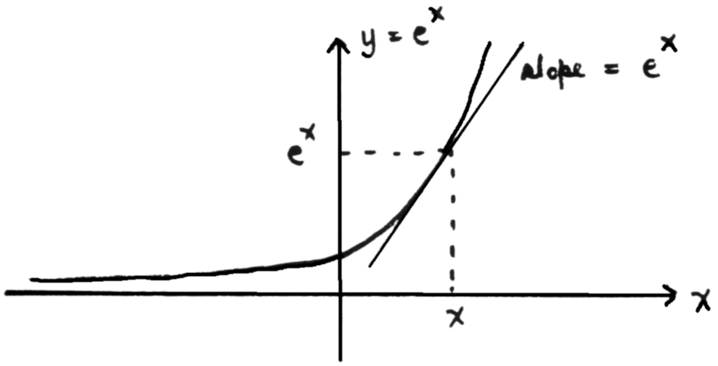
- First prove the result when x is a positive integer. (An easier proof than the one sketched above uses the product rule for differentiation.)
- Use the quotient rule for differentiation to extend the result to the negative integers.
- Use the formula for the derivative of an inverse function to extend the result to exponents of the form $\frac{1}{n}$.
- Write $x^{p/q} = (x^{1/q})^p$ to extend the result to all rational exponents.
- Use the exponential function to make sense of irrational exponents: $x^r = e^{r \ln x}$. (Here, we require x to be positive.) Differentiate to complete the proof.

DIFFERENTIATION TOOL

differentiating e^x

If $f(x) = e^x$, then $f'(x) = e^x$.

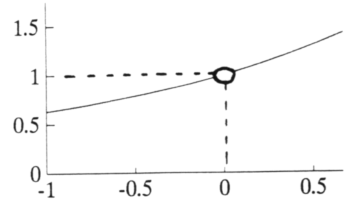
Thus, the *derivative of the exponential function is itself!* This is a property of the exponential function that is not shared by any other function. Make sure you understand what this fact is saying: if you look at any point on the graph of the function e^x, then the y-value of the point also tells you the slope of the tangent line to that point!



idea of proof

Let $f(x) = e^x$. Then:

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x e^h - e^x}{h}\\ &= \lim_{h\to 0} e^x\left(\frac{e^h-1}{h}\right)\end{aligned}$$



GRAPH OF $g(h) = \frac{e^h-1}{h}$

If it can be shown that $\lim_{h\to 0}\frac{e^h-1}{h} = 1$, then we can complete the proof:

$$\begin{aligned}\lim_{h\to 0} e^x\left(\frac{e^h-1}{h}\right) &= e^x\cdot\lim_{h\to 0}\frac{e^h-1}{h}\\ &= e^x(1) = e^x\end{aligned}$$

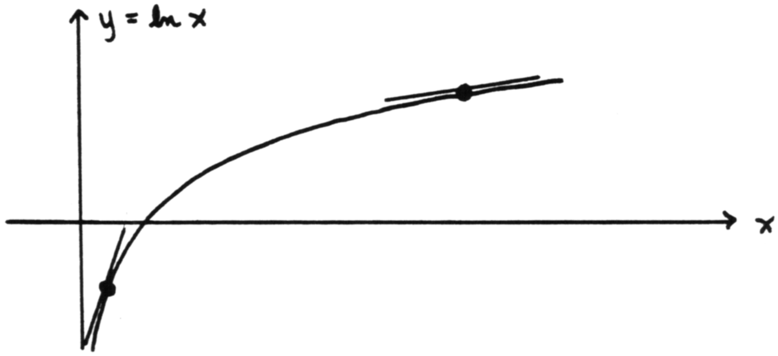
A graph of $g(h) := \frac{e^h-1}{h}$ for values of h close to 0 is shown, which illustrates the fact that $\lim_{h\to 0}\frac{e^h-1}{h} = 1$.

DIFFERENTIATION TOOL
differentiating $\ln x$

If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$.

the result is believable

Observe that this result *is* believable: when x is large, the slopes of tangent lines to the graph of $\ln x$ are small; and when x is close to 0, the slopes are large and positive.



EXAMPLE

To differentiate functions involving e^x and $\ln x$, it is often necessary to first rewrite the function, using properties of exponents and logs. These properties are reviewed in the Algebra Review at the end of this section.

Problem: Differentiate $f(x) = e^{2+x}$.

Solution: First write $f(x) = e^{2+x} = e^2 e^x$. Then,

$$f'(x) = e^2\frac{d}{dx}e^x = e^2e^x = e^{2+x}\ .$$

Another (easier) way to differentiate f will be possible after we study the Chain Rule for Differentiation.

EXAMPLE

Problem: Differentiate $g(x) = \ln 2x$.

Solution: First write $g(x) = \ln 2 + \ln x$. Then:

$$g'(x) = 0 + \frac{1}{x} = \frac{1}{x}$$

Another (easier) way to differentiate g will be possible after we study the Chain Rule for Differentiation.

EXERCISE 7

Differentiate each of the following functions. It will be necessary to first rewrite the functions, using properties of exponents and logarithms.

♣ 1. $f(x) = e^{x+5}$; interpret your result graphically.

♣ 2. $f(x) = \ln 7x$

♣ 3. Do you think that we have the necessary tools yet to differentiate $f(x) = e^{2x}$? Why or why not?

♣ 4. Do you think that we have the necessary tools yet to differentiate $g(x) = \ln(x+3)$? Why or why not?

A chart summarizing the tools developed in this section is given below:

DIFFERENTIATION TOOLS

prime notation	$\frac{d}{dx}$ operator
if $f(x) = k$, then $f'(x) = 0$	$\frac{d}{dx}(k) = 0$
$(kf)'(x) = k \cdot f'(x)$	$\frac{d}{dx}\big(kf(x)\big) = k \cdot f'(x)$
$(f+g)'(x) = f'(x) + g'(x)$	$\frac{d}{dx}\big(f(x)+g(x)\big) = f'(x) + g'(x)$
$(f-g)'(x) = f'(x) - g'(x)$	$\frac{d}{dx}\big(f(x)-g(x)\big) = f'(x) - g'(x)$
if $f(x) = x^n$, then $f'(x) = nx^{n-1}$	$\frac{d}{dx}x^n = nx^{n-1}$
if $f(x) = e^x$, then $f'(x) = e^x$	$\frac{d}{dx}(e^x) = e^x$
if $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$	$\frac{d}{dx}(\ln x) = \frac{1}{x}$

ALGEBRA REVIEW

radicals and fractional exponents, properties of logarithms

radicals

A *radical* is an expression of the form

$$\sqrt[n]{x}\ , \qquad (*)$$

for $n = 2, 3, 4, \ldots$.

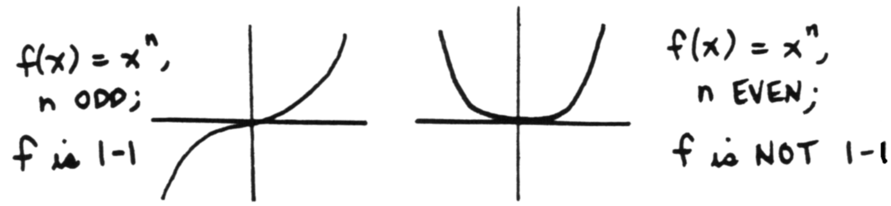
When $n = 2$, (*) is written more simply as $\sqrt{x}$, and is read as *the square root of x* .

When $n = 3$, $\sqrt[3]{x}$ is read as *the cube root of x* .

For $n \ge 4$, $\sqrt[n]{x}$ is read as *the n^{th} root of x* .

meaning of $\sqrt[n]{x}$

The purpose of radicals is to 'undo' exponents. That is, radicals provide a sort of inverse to the 'raise to a power' operation. Unfortunately, the 'raise to a power' functions $f(x) = x^n$ are only 1−1 if n is odd. When n is even, special considerations need to be made.



ODD roots

First consider $f(x) = x^3$. Here, f is 1−1, and its inverse is the cube root function, $f^{-1}(x) = \sqrt[3]{x}$. That is:

For all real numbers x and y:

$$y = \sqrt[3]{x} \iff y^3 = x$$

Rephrasing, y is the cube root of x if and only if y, when cubed, equals x.

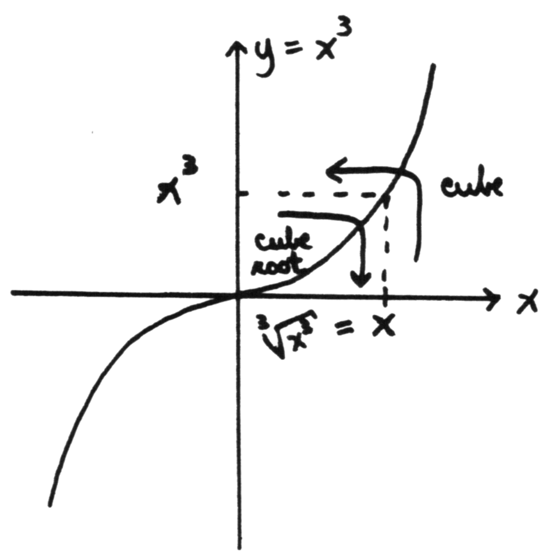
Thus, $\sqrt[3]{8} = 2$, since 2 is the unique number which, when cubed, equals 8.

Also, $\sqrt[3]{-8} = -2$, since -2 is the unique number which, when cubed, equals -8.

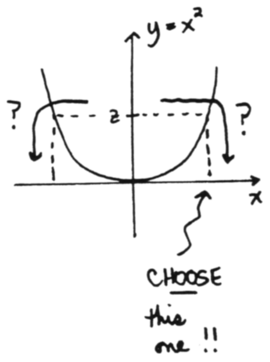
Indeed, for all real numbers x, and for $n = 3, 5, 7, 9, \ldots$,

$$\sqrt[n]{x^n} = x \ ,$$

since x is the unique real number which, when raised to an *odd* n^{th} power, equals x^n.



EVEN roots



When n is even, $f(x) = x^n$ is NOT 1−1. Consider, for example, $f(x) = x^2$. Here, (as for all even powers), $\mathcal{R}(f) = [0, \infty)$. Given $z \in \mathcal{R}(f)$, there are TWO inputs which, when squared, give z. *Mathematicians have agreed to choose the NONNEGATIVE number which works.* Precisely, we have:

For all $x \geq 0$ and for all real numbers y:

$$y = \sqrt{x} \quad \iff \quad y \geq 0 \ \text{ and } y^2 = x$$

That is, y is the square root of x if and only if y is nonnegative, and y, when squared, equals x.

Thus, $\sqrt{4} = 2$, since 2 is nonnegative, and $2^2 = 4$.

The expression $\sqrt{-4}$ is not defined, because there is NO real number, which when squared, equals -4.

What is $\sqrt{x^2}$? There are TWO real numbers which, when squared, give x^2: x and $-x$. We need to choose whichever is nonnegative. The absolute value comes to the rescue:

For all real numbers x:

$$\sqrt{x^2} = |x|$$

Indeed, for all nonnegative real numbers x, and for all $n = 2, 4, 6, 8, \ldots$, we have:

$$\sqrt[n]{x^n} = |x|$$

EXERCISE 8

practice with radicals

♣ 1. Consider this mathematical sentence:

For all real numbers x and y:

$$y = \sqrt[3]{x} \quad \iff \quad y^3 = x \qquad (*)$$

This sentence compares two 'component' sentences. What are they? What is (*) telling us that they have in common?

What is (*) telling us (if anything) when $y = 2$ and $x = 8$? How about when $y = -2$ and $x = 8$?

♣ 2. Consider this mathematical sentence:

For all $x \geq 0$ and for all real numbers y:

$$y = \sqrt{x} \quad \iff \quad y \geq 0 \ \text{ and } \ y^2 = x \qquad (**)$$

What two component sentences are being compared? What do they have in common?

What is (**) telling us (if anything) when $y = 2$ and $x = 4$? How about when $y = -2$ and $x = 4$?

Evaluate the following roots. Be sure to write complete mathematical sentences. State any necessary restrictions on x and y.

♣ 3. $\sqrt[5]{-32}$

♣ 4. $\sqrt[4]{(-2)^4}$

♣ 5. $\sqrt[6]{x^6}$

♣ 6. $\sqrt[9]{x^9}$

fractional exponent notation

When working with radicals in calculus, it is usually more convenient to use *fractional exponent notation* rather than *radical notation.*

Whenever $\sqrt[n]{x}$ *is defined, it can be alternately written as* $x^{\frac{1}{n}}$*.*

Thus:

- $\sqrt{5} = 5^{\frac{1}{2}}$
- $\sqrt[3]{x} = x^{1/3}$ for all real numbers x
- $\sqrt[4]{x} = x^{1/4}$ for all nonnegative real numbers x

Then, using properties of exponents (which are summarized below for your convenience), one can make sense of arbitrary rational exponents:

$$x^{\frac{p}{q}} = (x^p)^{\frac{1}{q}} = \sqrt[q]{x^p}$$

or

$$x^{\frac{p}{q}} = (x^{\frac{1}{q}})^p = (\sqrt[q]{x})^p ,$$

provided that both $\sqrt[q]{x^p}$ and $(\sqrt[q]{x})^p$ are defined. Use whichever representation is easiest for a given problem.

PROPERTIES OF EXPONENTS

Assume that a, b, n and m are restricted to values for which each expression is defined.

$a^m \cdot a^n = a^{m+n}$	(same base, multiplied, add exponents)
$\frac{a^m}{a^n} = a^{m-n}$	(same base, divided, subtract exponents)
$(a^m)^n = a^{mn}$	(power to a power, multiply exponents)
$(ab)^m = a^m b^m$	(product to a power, each factor gets raised to the power)
$(\frac{a}{b})^m = \frac{a^m}{b^m}$	(quotient to a power, both numerator and denominator get raised to the power)
$a^{-n} = \frac{1}{a^n}$	(definition of negative exponents)
$a^0 = 1$ for $a \neq 0$	(definition of zero exponent)

EXERCISE 9

♣ Convince yourself that each of these exponent laws 'makes sense'. Just look at special cases, where convenient.

For example, for positive integers m and n:

$$a^m \cdot a^n = \overbrace{(a \cdot \ldots \cdot a)}^{m \text{ factors of } a} \cdot \overbrace{(a \cdot \ldots \cdot a)}^{n \text{ factors of } a} = \overbrace{a^{m+n}}^{m+n \text{ factors of } a}$$

EXAMPLE
working with fractional exponents

Problem: Rewrite using fractional exponent notation. State any necessary restrictions on x and y. Where possible, write in two different ways.

1. $y = \sqrt[5]{x^3}$
2. $f(x) = \frac{\sqrt{x}\,\sqrt[3]{x^5}}{x}$

Solutions:

1. $y = \sqrt[5]{x^3} = (x^3)^{1/5} = x^{3\cdot\frac{1}{5}} = x^{3/5}$
2. Observe that $\mathcal{D}(f) = \{x \mid x > 0\}$. For such x:

$$\begin{aligned} f(x) &= \frac{\sqrt{x}\,\sqrt[3]{x^5}}{x} = \frac{x^{1/2}(x^5)^{1/3}}{x} \\ &= \frac{x^{1/2}x^{5/3}}{x} = \frac{x^{\frac{1}{2}+\frac{5}{3}}}{x} \\ &= \frac{x^{\frac{3}{6}+\frac{10}{6}}}{x} = \frac{x^{\frac{13}{6}}}{x^{\frac{6}{6}}} \\ &= x^{\frac{13}{6}-\frac{6}{6}} = x^{7/6} \\ &= (x^7)^{1/6} = \sqrt[6]{x^7} \end{aligned}$$

Alternately, if desired:

$$x^{7/6} = (x^{1/6})^7 = (\sqrt[6]{x})^7$$

All the steps were shown in the above display. You will probably be able to do a number of these steps in your head.

properties of $\ln x$

Next, some properties of logarithms are reviewed.

a precise view of functions

Whenever f is a function, then every input has a unique corresponding output. In other words, whenever two inputs are the same (and perhaps just have different names), then they must have the same output. Precisely, whenever f is a function with domain elements a and b:

$$a = b \quad \Longrightarrow \quad f(a) = f(b) \tag{1}$$

Thus, whenever the sentence '$a = b$' is true, so is the sentence '$f(a) = f(b)$'.

a precise view of a $1-1$ *function*

If f is, in addition, a $1-1$ function, then every output has a unique corresponding input. In other words, whenever two outputs are the same, then they must have come from the same input. Precisely, whenever f is a $1-1$ function with domain elements a and b:

$$f(a) = f(b) \quad \Longrightarrow \quad a = b \tag{2}$$

Thus, whenever the sentence '$f(a) = f(b)$' is true, so is the sentence '$a = b$'. Putting (1) and (2) together, whenever f is a $1-1$ function with domain elements a and b:

$$a = b \quad \Longleftrightarrow \quad f(a) = f(b)$$

Thus, if two inputs are the same, so are the corresponding outputs (the function condition); and whenever two outputs are the same, so are the corresponding inputs (the $1-1$ condition).

EXAMPLE

The function $f(x) = e^x$ is 1−1 and has domain $\mathbb{R}$. Thus, for all real numbers x and y:

$$x = y \iff e^x = e^y$$

The function $g(x) = \ln x$ is 1−1 and has as its domain the set of positive real numbers. Thus, for all positive real numbers x and y:

$$x = y \iff \ln x = \ln y$$

e^x and $\ln x$ are inverse functions

In addition, recall that e^x and $\ln x$ are inverse functions. Thus, a point (x, y) lies on the graph of $f(x) = e^x$ exactly when the point (y, x) lies on the graph of $g(x) = \ln x$. That is, for all $y > 0$ and for all real numbers x:

$$y = e^x \iff x = \ln y$$

We are now in a position to verify some important properties of logarithms, which are summarized below for convenience:

PROPERTIES OF LOGARITHMS

Assume that a and b are restricted to values for which each expression is defined

$$\ln(ab) = \ln a + \ln b$$
$$\ln \frac{a}{b} = \ln a - \ln b$$
$$\ln a^b = b \ln a$$

sample proof

The first equation says that *the log of a product is the sum of the logs.*

Here is its proof. The remaining proofs are left as exercises.

Let $a > 0$ and $b > 0$, so that all three expressions $\ln(ab)$, $\ln a$, and $\ln b$ are defined. Then:

$$\begin{aligned} y = \ln a + \ln b &\iff e^y = e^{\ln a + \ln b} && (e^x \text{ is a 1−1 function}) \\ &\iff e^y = e^{\ln a} e^{\ln b} && (\text{properties of exponents}) \\ &\iff e^y = ab && (e^x \text{ and } \ln x \text{ ‘undo’ each other}) \\ &\iff \ln e^y = \ln ab && (\ln x \text{ is a 1−1 function}) \\ &\iff y = \ln ab \end{aligned}$$

Thus, the sentences $y = \ln a + \ln b$ and $y = \ln ab$ always have the same truth values. That is, $\ln ab = \ln a + \ln b$.

EXERCISE 10

♣ 1. In words, what does

$$\ln \frac{a}{b} = \ln a - \ln b$$

say? Prove it. Be sure to justify every step of your proof.

♣ 2. Prove that:

$$\ln a^b = b \ln a$$

Be sure to write complete mathematical sentences, and justify every step of your proof.

QUICK QUIZ
sample questions

1. Differentiate $f(x) = \sqrt{x}$. Write the derivative using both prime notation, and Leibniz notation.
2. TRUE or FALSE: $\frac{d}{dx}(\frac{\pi\sqrt{2}}{7+\sqrt{3}}) = 0$
3. TRUE or FALSE: The slope of the tangent line to the graph of $y = x^3$ at the point $(2, 8)$ is 12. Show any work leading to your answer.
4. Expand $(a - b)^4$, using Pascal's Triangle.
5. Let $g(x) = e^x + \ln x$. Find $g'(x)$.

KEYWORDS
for this section

Prime notation for the derivative, Leibniz notation for the derivative, the operator $\frac{d}{dx}$*, the derivative of a constant, constants can be 'slid out' of the differentiation process, differentiating sums and differences, the Simple Power Rule for differentiation, Pascal's Triangle, differentiating* e^x *and* $\ln x$*, radicals and fractional exponent notation, properties of* $\ln x$*.*

END-OF-SECTION EXERCISES

♣ Differentiate the following functions. Feel free to use any tools developed in this section.

♣ 1. $f(x) = (2x + 1)^3$

♣ 2. $g(x) = \frac{\sqrt{x}+1}{\sqrt[7]{x}}$

♣ 3. $h(x) = \begin{cases} 3x^2 - 2x + 1 & x \ge 1 \\ 4x - 2 & x < 1 \end{cases}$
What is $\mathcal{D}(h)$?
What is $\mathcal{D}(h')$?

♣ 4. $h(x) = \begin{cases} 3x^2 - 2x + 1 & x \ge 1 \\ 3x - 1 & x < 1 \end{cases}$
What is $\mathcal{D}(h)$?
What is $\mathcal{D}(h')$?