1.5 Graphs

graphs

The word 'graph' always refers to some pictorial representation of information. Graphs are particularly helpful when there is a large amount of information (often infinite) to be understood.

In this section, we study graphs. The section is rather long, but most of the material should be review.

the graph of a sentence

The *graph of a sentence* (equation/inequality) is just a *picture of its solution set.* More correctly, the phrase usually refers to a *partial* picture of the solution set. The tool used to show this 'picture' depends on the nature of the solution set; whether it is a collection of *numbers*, or *pairs of numbers*, or, say, *triples of numbers.*

graphing sentences in one variable; use the real number line

Suppose you are asked to graph the equation in one variable, $x = 2$. For any such equation in one variable, the solution set is a collection of *numbers*, and the real number line can be used to display these numbers. Here, the solution set is $\{2\}$, since 2 is the only real number that is equal to 2. The graph is very uninteresting: one dot (at 2) on the real number line.

The graph of $x = 2$, viewed as an equation in one variable

You'll rarely be asked to graph simple equations in one variable like this: these types of equations usually have a small finite number of solutions, and a picture is not needed to understand this information.

EXERCISE 1

graphing sentences in one variable

Graph the following sentences in one variable:

♣ 1. $3x = x + x + x$

♣ 2. $x^2 < 0$

♣ 3. $x^2 \le 0$

♣ 4. $(x-3)(x+1) = 0$ (see Algebra Review—zero factor law)

♣ 5. $2x - 1 = 7$

♣ 6. $x^2 - 4x = 5$ (see Algebra Review—zero factor law)

♣ 7. $|x| = 2$

♣ 8. $|t| < 2$ and $t \ge 1$

graphing sentences in two variables; use the rectangular coordinate system

The graphs of sentences in two variables are usually much more interesting, because such sentences usually have an infinite number of members in their solution set.

For example, consider the two-variable equation $y = x$. Remember that a solution of this equation consists of a pair of numbers—a choice for x and a choice for y—that makes the equation true. Once a choice is made for x, the choice for y is uniquely determined, since y must equal x.

Here are some of the ordered pairs in the solution set of this equation: $(0, 0)$, (π, π), $(-\frac{2}{3}, -\frac{2}{3})$, and $(-2017.1, -2017.1)$. It is of course impossible to list everything in the solution set, but the solution set *can* be described compactly using set-builder notation: the solution set of the equation $y = x$ is

$$\{(x, y) \mid y = x\} = \{(x, x) \mid x \in \mathbb{R}\} \ .$$

The graph of this equation is then a (partial) 'picture' of all these ordered pairs. But how can we 'picture' an ordered pair? Answer: by using the *rectangular coordinate system*, discussed next.

DEFINITION

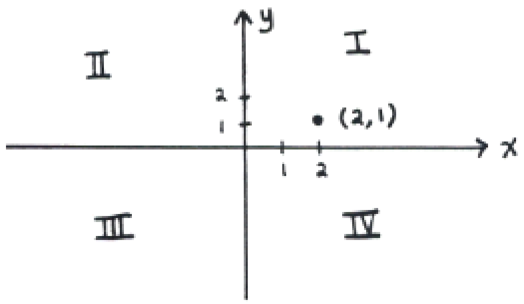
the rectangular coordinate system

origin

x-axis

y-axis

The device used for graphically representing ordered pairs is called the *rectangular coordinate system*, also commonly referred to as the *Cartesian plane* (named after the French mathematician René Descartes, 1596–1650). It is the plane formed by two intersecting perpendicular lines. Their point of intersection is called the *origin*. By convention, the horizontal line is called the *x-axis* and the vertical line the *y-axis*, with positive directions to the right and up, respectively. (Thus, yet another name commonly used for the Cartesian plane is the *xy-plane*.)



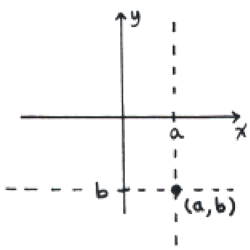
quadrant I
quadrant II
quadrant III
quadrant IV

The xy-plane is naturally divided into four quadrants, which are numbered as follows:

- quadrant I $= \{(x, y) \mid x > 0 \text{ and } y > 0\}$.
- quadrant II $= \{(x, y) \mid x < 0 \text{ and } y > 0\}$.
- quadrant III $= \{(x, y) \mid x < 0 \text{ and } y < 0\}$.
- quadrant IV $= \{(x, y) \mid x > 0 \text{ and } y < 0\}$.

Observe that the axes are not part of any quadrant.

every ordered pair corresponds to a unique point in the xy-plane



Every ordered pair (a, b) corresponds to a unique point in the xy-plane in the following way:

- go to 'a' on the x-axis; draw a vertical line through this point;
- go to 'b' on the y-axis; draw a horizontal line through this point;
- the unique point where these two lines intersect is the point associated with the ordered pair (a, b).

Here's a slightly less precise, but perhaps easier way to find the point (a, b): start at the origin, move 'a' units in the x-direction, then 'b' units in the y-direction. (If a is positive, move to the right; if a is negative, move to the left. If b is positive, move up; if b is negative, move down.)

♣ Could we say this instead? *Start at the origin, move 'b' units in the y-direction, then 'a' units in the x direction.*

we will freely interchange the words 'point' and 'ordered pair'

Observe that every ordered pair corresponds to precisely one point in the plane; and every point in the plane corresponds to precisely one ordered pair. Thus, we can freely interchange the words 'ordered pair' and 'point' without confusion.

*Why the name '**ordered** pair'?*

NOTE: If $x \neq y$, then (x, y) is a different point than (y, x). Hence the name '**ordered** pair'!

EXERCISE 2

♣ 1. Plot the points $(1,3)$, $(-1,-5)$, $(0,-\pi)$, $(-\sqrt{2},0)$ on a rectangular coordinate system. What quadrant (if any) is each point in?

♣ 2. Plot several points in the solution set of the equation $y = x$. See a pattern forming?

DEFINITION

graph of a sentence in 2 variables

The *graph* of a sentence in 2 variables is a (partial) picture of its solution set; that is, it is the set of all ordered pairs (x, y) that satisfy the sentence, displayed on a rectangular coordinate system.

A graph portrays an infinite number of solution points in an organized, easy-to-analyze fashion. In many cases, one of the variables is allowed to range over an infinite interval of real numbers, so that it is impossible to show the entire graph. In such instances, one usually shows a representative part of the graph, or enough of the graph to capture everything of interest. This is illustrated in the next examples.

EXAMPLE

graph of $x = 2$, viewed as an equation in two variables, $x + 0y = 2$

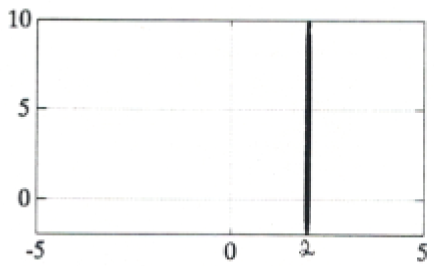
Suppose you are asked to graph the equation $x = 2$. Out of context, you should rightfully be confused: are you to treat this as an equation in one variable (x), or two variables (x and y, say $x + 0y = 2$)? The answer is important. As was seen earlier, if $x = 2$ is treated as an equation in one variable, the graph is boring—a single point at 2 on the real number line. However, treated as an equation in two variables, its graph is

$$\{(x,y) \mid x = 2,\ y \in \mathbb{R}\} = \{(2,y) \mid y \in \mathbb{R}\}\ .$$

Thus, the graph is the vertical line shown below.

Note: A comma is sometimes used in mathematics to mean the mathematical word 'AND'. Thus,

$$\{(x,y) \mid x = 2,\ y \in \mathbb{R}\} = \{(x,y) \mid x = 2 \ \text{ AND } \ y \in \mathbb{R}\}\ .$$



The graph of $x = 2$, viewed as an equation in two variables.

EXAMPLE

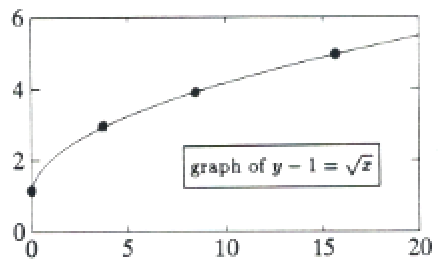
graphing a simple equation in 2 variables

Problem: Graph the equation $y - 1 = \sqrt{x}$.

Solution: Observe that here, y can easily be solved for x by adding 1 to both sides and obtaining the equivalent equation $y = \sqrt{x} + 1$. Thus, it is easy to choose (allowable) values for x, and compute the corresponding value of y:

x	$y = 1 + \sqrt{x}$
0	1
4	3
9	4
16	5
25	6

Note that x was chosen so that the corresponding y values were easy to compute. Plotting these points appears to illustrate a pattern; the graph is completed by drawing a smooth curve through the sample points. (Does this last step make you a bit uneasy? More on this in a moment.) Observe that the resulting graph is only a partial picture of all the ordered pairs that make the equation true.



EXERCISE 3 ♣ Graph the equation $x - 1 = \sqrt{y}$. Compare your graph with the one in the previous example.

EXAMPLE
graphing an inequality in 2 variables

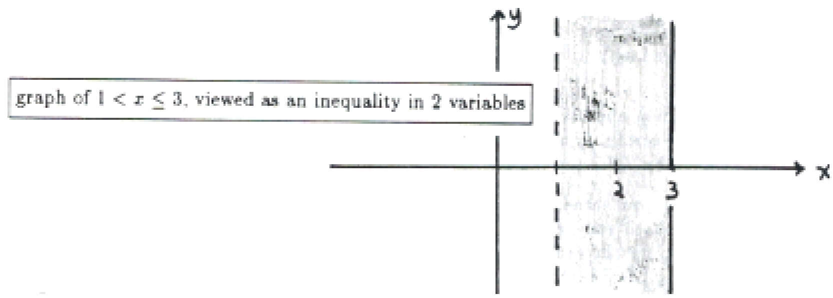
Problem: Graph the compound inequality $1 < x \leq 3$, viewed as an inequality in 2 variables (say, $1 < x + 0y \leq 3$).

Solution: The solution set is

$$\{(x, y) \,|\, 1 < x \leq 3, \quad y \in \mathbb{R}\} .$$

Thus, we seek all points with x values between 1 and 3 (not including 1, including 3). The points can have *any* y-values.

The graph is shown below. Note that a *solid* line means that points are to be included; a *dashed* line means that points are *not* to be included.



EXAMPLE
graphing an inequality in 2 variables

Problem: Graph the inequality $y > x$.

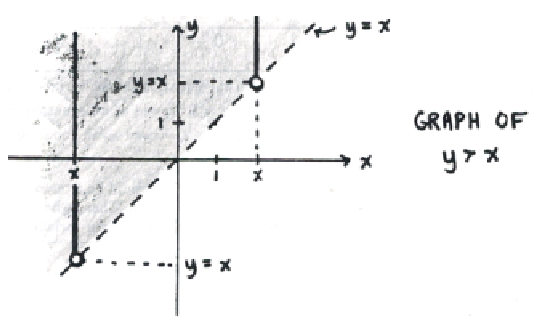
Solution: The solution set is

$$\{(x, y) \,|\, x \in \mathbb{R},\; y > x\} .$$

Thus, we seek all points (x, y) with the property that their y-value is *greater than* their x-value. For example, the point $(1, 1.1)$ satisfies this property, but the point $(-3, -3.1)$ does not.

The easiest way to obtain this graph is to first graph the *boundary*, $y = x$. We don't want these points, whose y-values *equal* their x-values, so the line $y = x$ is dashed.

Now, choose any value of x. Corresponding to this x-value, we want all points with y-values *greater than* x. Thus, we want the points that lie *above* the line. Letting x vary over all real numbers, the desired graph consists of all the points above the line $y = x$.



general scheme for graphing an equation in 2 variables

Given an equation in 2 variables, our basic goal is to gain a good understanding of what the solution pairs look like. If we are able to solve the equation for one of the variables, say y, in terms of the remaining variable—that is, get it in the form

$$y = < \text{stuff involving } x > \ ,$$

then solution points are easily generated: merely choose allowable values for x, and calculate the corresponding values of y.

conjecture

If these points are plotted, a pattern may be displayed, leading to a conjecture (educated guess) about the form of the graph. However, *plotting points is extremely inefficient and not foolproof*, and should only be used in connection with other methods. Plotting several points, however, is always a good way to begin: it can help to confirm one's belief about the nature of a graph, or catch mistakes.

classifying an equation

One common approach to graphing an equation is to *classify the equation* as being of a certain type. Then, use information about this known type to graph the equation. The approach is illustrated in the next example.

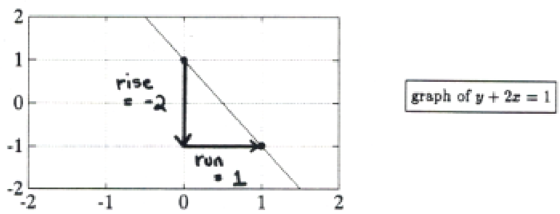
EXAMPLE

lines;

$ax + by = c$

In algebra, you learned that every equation of the form $ax+by = c$, when a and b are not both zero, graphs as a line in the xy-plane. This class of equations (one for every allowable choice of a, b and c) is known as *the linear equations in x and y*. Once an equation has been identified as linear, only two points need to be plotted to obtain the graph. Or, one can rewrite the equation in an equivalent form that is easier to work with.

For example, consider the equation $y+2x = 1$. This is equivalent to $y = -2x+1$, which is now in slope-intercept form, $y = mx + b$ (see Algebra Review—lines). Thus, one can 'read off' that the line crosses the y-axis at 1, and has a slope of $-2 = \frac{-2}{1} = \frac{\text{rise}}{\text{run}}$. Using this information, the line is easily graphed.



EXERCISE 4

♣ 1. Graph the equation in two variables, $x - 3y + 5 = 0$.

♣ 2. Graph $y = 3$, viewed as an equation in two variables.

♣ 3. Graph $|y| > 2$, viewed as an equation in two variables.

♣ 4. Graph $y < -x$.

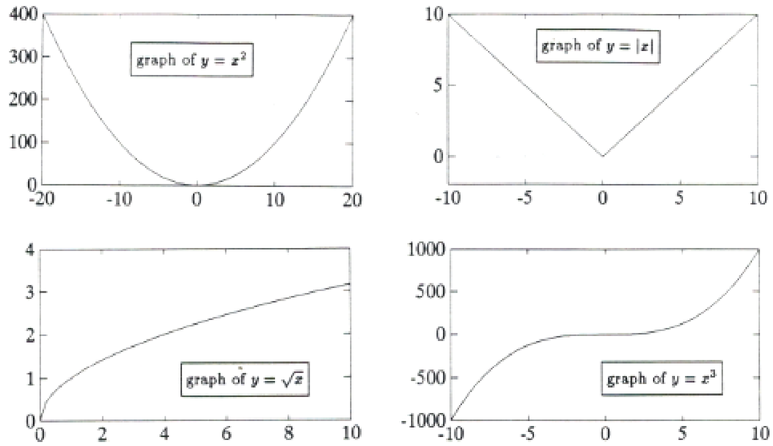
♣ 5. Think about what would be a reasonable way to 'picture' ordered triples (x, y, z) of real numbers. Then, what would the graph of $x = 2$ look like, viewed as an equation in three variables, $x + 0y + 0z = 2$?

using calculus to graph

The techniques of calculus give us extremely powerful tools for graphing many equations in 2 variables. These techniques will be discussed later on in this text.

common graphs that you should know

You should be familiar with all the following common graphs:



common variations on these graphs

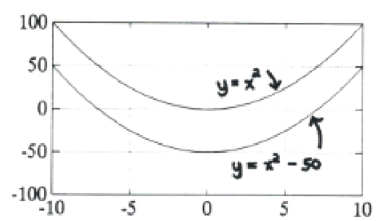
Once the shapes of these basic graphs are known, it is easy to obtain some slight modifications. For example, consider the graph of $y = x^2 - 50$. One must form a picture of its solution set,

$$\{(x, y) \mid y = x^2 - 50\} = \{(x, x^2 - 50) \mid x \in \mathbb{R}\}\ .$$

How does this relate to the graph of $y = x^2$, whose solution set is

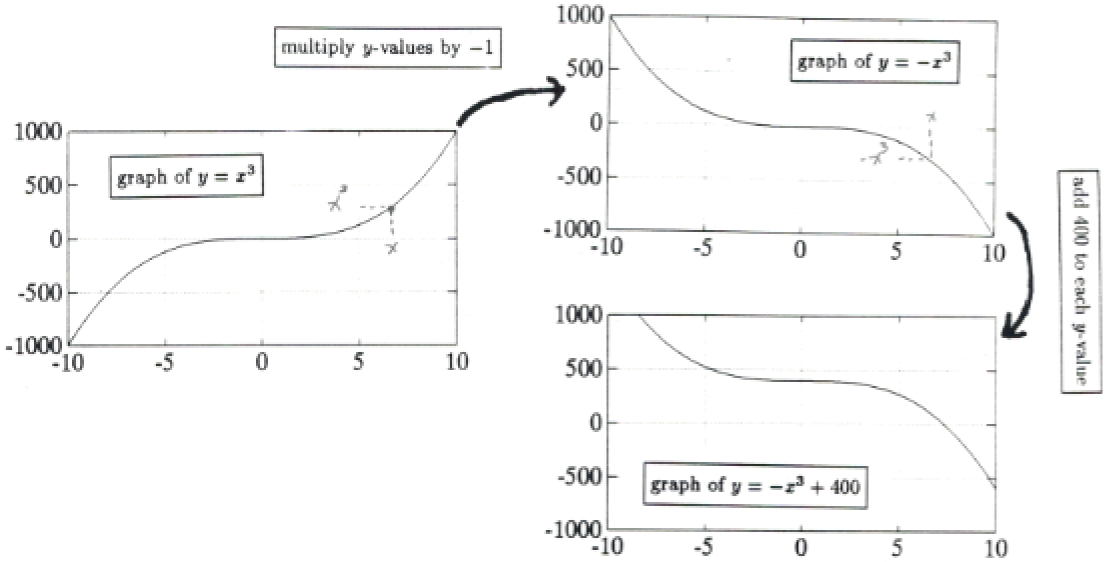
$$\{(x, x^2) \mid x \in \mathbb{R}\}\ ?$$

Each y-value has been reduced by 50; hence the graph of $y = x^2$ must be shifted down 50 units to obtain the graph of $y = x^2 - 50$.



EXAMPLE
'building up' a graph from simpler pieces

The sequence of graphs below illustrates how one can easily obtain the graph of $y = -x^3 + 400$.



EXERCISE 5

Graph the following equations, by building them up from simpler pieces.

♣ 1. $y = -x^2 + 1$

♣ 2. $|x| + y = 3$

♣ 3. $\sqrt{x} - 4 + y = 0$

ALGEBRA REVIEW

zero factor law, mathematical word 'or', lines

THEOREM
the zero factor law

THEOREM (The Zero Factor Law). For all real numbers a and b,

$$ab = 0 \iff a = 0 \text{ or } b = 0 \ .$$

translation of the 'zero factor law'

Here is how an algebra instructor might translate this theorem for students: *Whenever a product of real numbers equals zero, at least one of the factors must be zero.*

★ The theorem actually says more than this, but the instructor is paraphrasing the most useful part of the result.

One goal of this course is that *you* become able to do the 'translating' yourself. To this end, you must first understand the meaning of the mathematical word 'or'.

the mathematical word 'or'

By definition, a mathematical sentence of the form

$$A \text{ or } B$$

is true when A is true, or B is true, or BOTH A and B are true.

This information is summarized in the truth table below:

A	B	A OR B
T	T	T
T	F	T
F	T	T
F	F	F

EXAMPLE

truth of 'or' sentences

For example, the mathematical sentence

$$2 = 1 \text{ or } 3 = 2 + 1$$

is true because '$3 = 2 + 1$' is true. The mathematical sentence

$$\sqrt{9} = 3 \text{ or } -3^2 = 9$$

is true because '$\sqrt{9} = 3$' is true (even though '$-3^2 = 9$' is false). The mathematical sentence

$$(-3)^2 = -9 \text{ or } \sqrt{(-4)^2} = -4$$

is false because both component sentences are false.

The sentence '$x = 3$ or $x = 5$' is conditional. Its solution set is $\{3, 5\}$. For all other choices of x, it is false.

CAUTION: the English word 'or' versus the mathematical word 'or'

CAUTION: there's a slight difference in the English and mathematical uses of the word 'or'.

If you say to a friend, "I'm going to study math or English tonight," you probably mean that you'll study math, or English, but NOT BOTH.

However, when a mathematician makes a true statement 'A or B', this means that A is true, or B is true, or BOTH A and B are true.

EXERCISE 6

the mathematical word 'or'

Classify each sentence as (always) TRUE, (always) FALSE, or CONDITIONAL:

♣ 1. $1 < 1$ or $1 > 1$

♣ 2. $1 \le 1$ or $1 > 1$

♣ 3. $|x| > 0$ or $|x| = 0$

♣ 4. $x = 3$ or $x = -3$

♣ 5. $\sqrt{t^2} = t$ or $\sqrt{t^2} = -t$

return to the 'zero factor law'

Now, return to the zero factor law. What is it telling us? The theorem compares two mathematical sentences: the sentence '$ab = 0$' and the sentence '$a = 0$ or $b = 0$'. Since these two sentences are equivalent, they always have the same truth values. Therefore, the two sentences can be used interchangeably.

For example, choosing $a = 3$ and $b = 0$, the equation '$ab = 0$' becomes '$3 \cdot 0 = 0$', which is true; and the sentence '$a = 0$ or $b = 0$' becomes '$3 = 0$ or $0 = 0$', which is also true.

♣ What is the theorem telling us if $a = 1$ and $b = 2$?

typical use of the zero factor law

Here's how the zero factor law is typically used. Suppose you are asked to solve the equation $x^2 - x - 6 = 0$. This equation cannot be solved by inspection. So it is transformed into an *equivalent* equation that *can* be solved by inspection, as follows. First, factor the left-hand side, and then use the zero factor law:

$$\begin{aligned} x^2 - x - 6 = 0 &\iff (x-3)(x+2) = 0 \\ &\iff x - 3 = 0 \ \text{ or } \ x + 2 = 0 \\ &\iff x = 3 \ \text{ or } \ x = -2 \ . \end{aligned}$$

Here, the zero factor law was applied with 'a' equal to '$x - 3$' and 'b' equal to '$x + 2$'. The last equation can be solved by inspection, and has solution set $\{3, -2\}$. Therefore, the equation $x^2 - x - 6 = 0$ also has solution set $\{3, -2\}$ (check this).

EXERCISE 7
using the zero factor law

Solve the following sentences. Use the zero factor law. Be sure to write complete mathematical sentences.

♣ 1. $x(5x - 3) = 0$

♣ 2. $x^2 - x = 12$

♣ 3. $(3x - 2)^2 - 16 = 0$

lines

Every line in the plane is uniquely determined by two distinct (different) points on the line. And every two distinct points uniquely determine a line. The information on lines included here should be a review; it is merely included for your convenience.

horizontal and vertical lines

The points on a horizontal line all have the same y-values; hence horizontal lines are all of the form $y = k$, for a real number k.

The points on a vertical line all have the same x-values; hence vertical lines are all of the form $x = k$, for a real number k.

EXERCISE 8

Graph each sentence. Interpret each as a sentence in two variables, x and y.

♣ 1. $3x = 4$

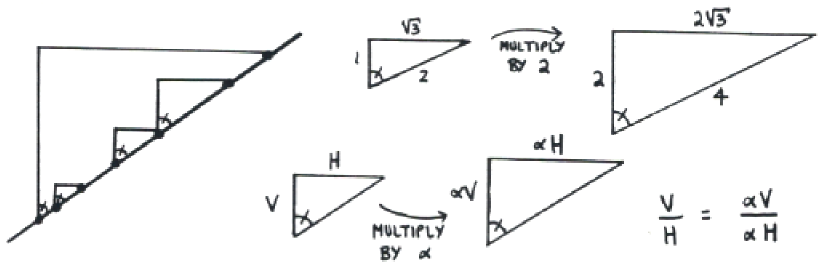
♣ 2. $5 = 4 - y$

♣ 3. $x = 4$ or $y = -1$

♣ 4. $x = 4$ and $y = -1$

non-vertical, non-horizontal lines

Non-vertical, non-horizontal lines have a beautiful property: no matter what two points are chosen on the line, the right triangles formed using the line as the hypotenuse *all have the same angles* (see the sketch below). Such triangles are called *similar triangles.* By appropriately 'magnifying' a triangle (that is, by multiplying *each side* by the same number), a triangle can be made to coincide with any similar triangle. This is illustrated in the sketch below.



slope of a line

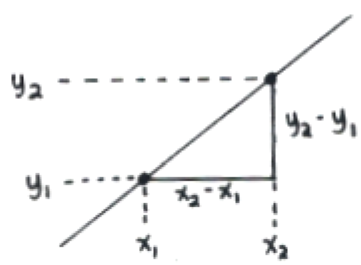
An important consequence of this fact is that the ratio of 'rise' (vertical travel) over 'run' (horizontal travel) in traveling from one point to another on the line, does NOT depend on which two points are used! This ratio is called the SLOPE OF THE LINE.

a convenient formula for the slope

Letting (x_1, y_1) and (x_2, y_2) be any two distinct points on a non-vertical line, the slope m of the line is found by:

$$m := \frac{y_2 - y_1}{x_2 - x_1} \ .$$

Note that the slope of a horizontal line is zero; the slope of a vertical line is undefined. ♣ WHY?



EXAMPLE
finding the slopes of lines

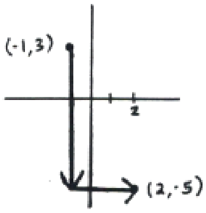
Problem: Find the slope of the line through the points $(-1,3)$ and $(2,-5)$.

Solution #1. Letting $(x_1,y_1)=(-1,3)$ and $(x_2,y_2)=(2,-5)$, we get

$$m=\frac{-5-3}{2-(-1)}=\frac{-8}{3}=-\frac{8}{3}\ .$$

Solution #2. Letting $(x_1,y_1)=(2,-5)$ and $(x_2,y_2)=(-1,3)$, we get

$$m=\frac{3-(-5)}{-1-2}=\frac{8}{-3}=-\frac{8}{3}\ .$$

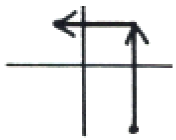


Solution #3. Traveling from the point $(-1,3)$ to the point $(2,-5)$ via the rules: 'rise first, then run', we obtain

$$m=\frac{\text{'down 8'}}{\text{'right 3'}}=\frac{-8}{3}=-\frac{8}{3}\ .$$

Solution #4. Traveling from the point $(2,-5)$ to the point $(-1,3)$ via the rules: 'rise first, then run', we obtain

$$m=\frac{\text{'up 8'}}{\text{'left 3'}}=\frac{8}{-3}=-\frac{8}{3}\ .$$



EXERCISE 9
finding slopes of lines

Find the slope of the line through each pair of points. If the slope is undefined, so state. Use several different approaches.

♣ 1. $(3,-2)$, $(-1,5)$

♣ 2. $(a,3)$, $(a,-1)$ (Here, a is any real number.)

♣ 3. $(-2,b)$, $(3,b)$ (Here, b is any real number.)

equations of lines;

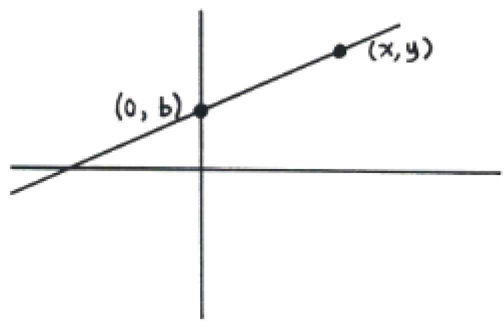
$y = mx + b$;

slope-intercept form

Every non-vertical line must cross the y-axis at exactly one point; call this point $(0, b)$. Let m denote the slope of the line. Then, if (x, y) is ANY other point on the line, then $x \neq 0$ (♣ Why?), so

$$\begin{aligned} m = \frac{y-b}{x-0} \quad &\iff \quad y - b = mx \\ &\iff \quad y = mx + b \ . \end{aligned}$$

That is, the solution set of the equation $y = mx + b$ is precisely the points on the line with slope m, that crosses the y-axis at b. The equation $y = mx + b$ is thus appropriately called the *slope-y-intercept form of a line.*



standard form of a line;

$ax + by = c$

If you stop and think a moment, you'll see that every equation of the form $ax + by = c$ (when a and b are not both zero), graphs as a line in the plane.

The set of all equations that can be written in the form $ax + by = c$, where a and b are not both zero, are called the *linear equations in* 2 *variables.* This is certainly a reasonable name, due to the previous observation!

EXAMPLE
graphing a line

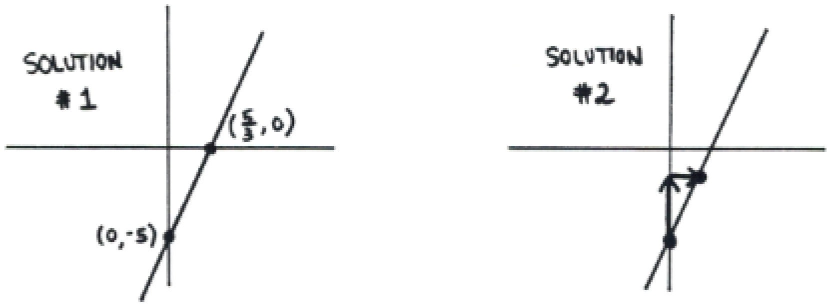
Problem: Graph $3x - y = 5$.

Solution #1: Recognize that the equation is linear in x and y; thus it graphs as a line. Plot ANY two points to graph the line! (So, choose two EASY points!)

When $x = 0$, we have $3(0) - y = 5 \iff y = -5$. Thus, the point $(0, -5)$ is on the graph.

When $y = 0$, we have $3x - 0 = 5 \iff x = \frac{5}{3}$. Thus, the point $(\frac{5}{3}, 0)$ is on the graph.

Plot these two points, and sketch the line through them.



Solution #2: Put the equation in $y = mx + b$ form, and read off b and m:

$$
\begin{aligned}
3x - y = 5 &\iff -y = 5 - 3x \\
&\iff y = 3x - 5 \ .
\end{aligned}
$$

The line crosses the y-axis at -5, and has a slope of 3:

$$m = 3 = \frac{3}{1} = \frac{\text{rise}}{\text{run}} \ .$$

Sketch the line.

EXERCISE 10
graphing

Graph the following sentences. Take two different approaches in each case.

♣ 1. $-5x = y + 4$

♣ 2. $x + y = 1$ or $x = 1$ or $y = 1$

QUICK QUIZ
sample questions

1. Graph the equation $x = 3$, viewed as an equation in 1 variable; viewed as an equation in 2 variables.
2. Graph the inequality $x < 3$, viewed as an inequality in 1 variable; viewed as an inequality in 2 variables.
3. Graph the equation $y - x^2 + 1 = 0$.
4. Graph the inequality $y \leq 2x$.
5. TRUE or FALSE: '$3 < 3$ or $3 \geq 3$'.
6. Find the value(s) of x for which the following sentence is TRUE; show them on a number line: '$x \geq 3$ or $x < -1$'.

KEYWORDS *for this section*	*Graphs, graph of a sentence, graphing sentences in one variable, graphing sentences in two variables, rectangular coordinate system (origin, x-axis, y-axis, quadrants I, II, III, IV), the zero factor law, the mathematical word 'or', conjecture, linear equations in* 2 *variables, graphing lines.*

END-OF-SECTION EXERCISES

Graph each of the following sentences in one variable. (Show the solution sets on a number line.)

1. $x = \pi$
2. $x - 3 = 0$
3. $|x| = 2$
4. $|x| \ge 2$
5. $3x < -2$
6. $2x - 5 > 3$
7. $x = 0$ or $|x| = 1$
8. $x = 1$ and $|x| = 1$
9. $x = 1$ or $|x| = 1$
10. $2x = 1$ and $2x \ne 1$
11. $|3x + 1| = 7$
12. $|3x + 1| < 7$

Graph each of the following sentences in two variables (x and y). (Show the solution sets in the xy-plane.)

13. $x + y = 2$
14. $y - 4x = -3$
15. $x = 1$ or $y = -2$
16. $x = 1$ and $y = -2$
17. $|y| = 1$
18. $|x| = 2$
19. $|x + y| = 1$ (Hint: For $a \ge 0$, $|z| = a \iff z = a$ or $z = -a$.)
20. $|x + y| < 1$ (Hint: For $a > 0$, $|z| < a \iff -a < z < a$.)