1.4 Mathematical Equivalence

Introduction

In this section, the idea of *mathematical equivalence* is introduced. Whereas the '=' sign gives us a way to compare mathematical *expressions*, the idea of 'being equivalent' gives us a way to compare mathematical *sentences*.

a motivating example

For motivation, consider the two mathematical sentences: '$2x - 3 = 0$' and '$x = \frac{3}{2}$'. They certainly *look* different. But in one very important way, they are the same: no matter what real number is chosen for the variable x, these two sentences *always have the same truth values*.

For example, choose x to be $\frac{3}{2}$.

Substitution into '$2x - 3 = 0$' yields the TRUE sentence '$2(\frac{3}{2}) - 3 = 0$'.

Substitution into '$x = \frac{3}{2}$' yields the TRUE sentence '$\frac{3}{2} = \frac{3}{2}$'.

Next, choose x to be, say, 5.

Substitution into '$2x - 3 = 0$' yields the FALSE sentence '$2(5) - 3 = 0$'.

Substitution into '$x = \frac{3}{2}$' yields the FALSE sentence '$5 = \frac{3}{2}$'.

No matter what real number is chosen for x, these two sentences will ALWAYS have the same truth values. Indeed, '$2x - 3 = 0$' is true when x is $\frac{3}{2}$, and false otherwise. Also, '$x = \frac{3}{2}$' is true when x is $\frac{3}{2}$, and false otherwise.

sentences that always have the same truth values can be used interchangeably

When two mathematical sentences always have the same truth values, then they can be used *interchangeably*, and a mathematician will use whichever sentence is easiest for a given situation.

The mathematical verb that is used to compare sentences with the same truth values is: 'is equivalent to'. Thus, it is correct to say that '$2x - 3 = 0$ is equivalent to $x = \frac{3}{2}$'.

the 'implied domain' of a sentence

Sentences naturally have a largest set of 'choices' for which the sentence is defined.

For example, the sentence '$\frac{1}{x} = 1$' is only defined for nonzero real numbers, since division by zero is undefined.

The sentence '$\sqrt{x} = 3$' is only defined for nonnegative real numbers.

The largest set of choices for which a sentence is defined will be referred to as the '*implied domain*' of the sentence, or more simply, the '*domain*' of the sentence. (Something is 'implied' if it is not explicitly stated, but merely understood.) Remember that, in this course, we are only 'choosing from' the real numbers. This idea is explored further in the next example.

EXAMPLE

finding 'implied domains' of sentences

Problem: Find the implied domain for each of the following sentences:

1. $\frac{1}{x(y-1)} = 2$
2. $\sqrt{x} > 0$
3. $\sqrt[3]{x} = -2$
4. $ax + by + c = 0$

Solution:

1. The expression '$\frac{1}{x(y-1)}$' is not defined if x is 0, or if $y = 1$. Thus, the implied domain of the sentence is $\{(x, y) \,|\, x \neq 0 \text{ and } y \neq 1\}$.

2. The expression '$\sqrt{x}$' is only defined for nonnegative real numbers x. Thus, the implied domain of the sentence is $\{x \,|\, x \geq 0\}$.

3. The expression '$\sqrt[3]{x}$' makes sense for all real numbers x. The implied domain of the sentence is $\mathbb{R}$.

4. In the sentence '$ax + by + c = 0$', convention dictates that only the x and y are variables; a, b and c are constants. This sentence is defined for all real numbers x and y; thus, the implied domain is $\{(x, y) \,|\, x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$.

EXERCISE 1

finding implied domains

Find the implied domain for each sentence. Write your answers using correct set notation.

♣ 1. $\frac{x}{y} = 1$

♣ 2. $ax + b = 0$

♣ 3. $\sqrt{x^2} - 4 = 0$

♣ 4. $\sqrt{x^3} = 2$

♣ 5. $\sqrt[4]{xy} - x = y - 5$. (Here, you may want to merely *shade* the allowable choices for (x, y) in the xy-plane.)

DEFINITION

equivalent sentences

Two mathematical sentences (with the same domains) are *equivalent* if they always have precisely the same truth values. That is, no matter what choice of variable(s) is made from the domain, if one sentence is true, so is the other; and if one sentence is false, so is the other.

EXAMPLE

equivalent sentences

The sentences $x + 1 = 0$ and $x = -1$ are equivalent. Each sentence has domain $\mathbb{R}$, because each is defined for all real numbers.

The sentence '$x + 1 = 0$' is true exactly when x is -1, and false otherwise.

The sentence '$x = -1$' is true exactly when x is -1, and false otherwise.

the symbol '$\iff$'

The symbol '$\iff$' is used by mathematicians to say that two sentences are equivalent. Thus, the sentence

$$x + 1 = 0 \quad \iff \quad x = -1$$

(read as '$x + 1 = 0$' is equivalent to '$x = -1$') means that the two component sentences being compared *always have the same truth values.*

★★

Experienced mathematicians realize that the sentence

$$x + 1 = 0 \iff x = -1$$

is really an implicit generalization:

$$\text{For all } x,\ x + 1 = 0 \iff x = -1\ .$$

The quantifier 'For all' is addressed later on in the text.

The precise definition of the connective ' $\iff$ ' is given by the following truth table:

A	B	A ⟺ B
T	T	T
T	F	F
F	T	F
F	F	T

determining if two sentences are equivalent, by comparing their solution sets

Suppose that two sentences have the same domains. If these two sentences have the same solution sets, then they must be TRUE at exactly the same time. Therefore, they must also be FALSE at exactly the same time.

This observation allows us to determine if two sentences (with the same domain) are *equivalent* by comparing their solution sets.

EXAMPLE

sentences that are NOT equivalent

The equations $x^2 = 9$ and $x = 3$ are *not* equivalent. (Remember that sentences using the verb '=' are called *equations*.) They both have domain $\mathbb{R}$, since both are defined for all real numbers. However, the first has solution set $\{3, -3\}$, whereas the second has solution set $\{3\}$.

That is, choose x to be -3. For this choice, the sentence '$x^2 = 9$' becomes '$(-3)^2 = 9$', which is TRUE; whereas the sentence '$x = 3$' becomes '$-3 = 3$', which is FALSE. The sentences do NOT always have the same truth value. They CANNOT be used interchangeably.

showing that two sentences are NOT equivalent

The previous example points out that to show that two sentences are NOT equivalent, we need only exhibit ONE choice of variable(s) for which the sentences have different truth values.

For example, the equations '$x = 0$' and '$x(x-1) = 0$' are NOT equivalent. Choosing x to be 1, the first sentence is false, but the second is true.

EXAMPLE

sentences with different domains

The sentences '$\frac{1}{x} = 1$' and '$x = 1$' have different domains. The first is defined only for nonzero real numbers; the second for *all* real numbers.

However, these sentences ARE very much alike: the first is undefined when x is 0, true when x is 1, and false otherwise. The second is true when x is 1, and false otherwise. As long as we restrict ourselves to choices for x for which BOTH sentences make sense, then they do act exactly the same.

For now, however, we will only compare sentences that have the SAME domains.

EXERCISE 2

Check that each sentence in a given pair has the same implied domain. Then, decide if the sentences are equivalent, or not.

♣ 1. $2x - 4 = 0,\ \ 3x - 6 = 0$

♣ 2. $x^2 - 16 = 0,\ \ x = 4$

♣ 3. $|x| = 3,\ \ x = 3$

♣ 4. $|x| = 3$, '$x = 3$ and $x = -3$' (Careful!)

♣ 5. $x > 0,\ \ 2x > 0$

♣ 6. $x > 0,\ \ -x < 0$

♣ 7. $x^2 \geq 0,\ \ x \geq 0$

♣ 8. $1 < x \leq 3$, '$x > 1$ and $x \leq 3$'

EXAMPLE

solving by inspection

The equations $2x + 3 = 0$ and $x = -\frac{3}{2}$ are equivalent; they have the same domains, and the same solution sets. Note that the second equation is 'simpler' than the first, in the sense that *it can be solved 'by inspection'*. That is, looking at the equation $x = -\frac{3}{2}$, it is *immediate* what makes it true: there is only one real number that is equal to $-\frac{3}{2}$.

EXERCISE 3

Which of the following sentences (if any) would you say can be solved '*by inspection*'?

♣ 1. $7x - 3 = 0$

♣ 2. $x^2 \geq 0$

♣ 3. $(x + 1)^2 \geq 0$

♣ 4. $x = -0.2$

♣ 5. $x < 0$

♣ 6. $x^3 < 0$

solving an equation

The goal in solving an equation is to transform the original (harder) equation into an *equivalent* one that can be solved easily. However, in this "transforming" process, we must be sure that we do *not* change the solution set! Thus we are interested in answering the question: *What can be done to an equation that does not alter its solution set*? The next two theorems answer this question:

THEOREM (Form A)

Let a, b and c be real numbers. Then:

$$a = b \iff a + c = b + c$$

If $c \neq 0$, then:

$$a = b \iff ac = bc$$

THEOREM (Form B)

Adding the same number to both sides of an equation does not change its solution set.

Subtracting the same number from both sides of an equation does not change its solution set.

Multiplying both sides of an equation by any nonzero number does not change its solution set.

Dividing both sides of an equation by any nonzero number does not change its solution set.

interpreting this theorem

Both of these theorems say exactly the same thing. Form A is the way a mathematician would give the answer, and this is the form that would appear in most textbooks.

Form B is the *translation* of Form A that an instructor makes so that the students can understand it. Most beginning students have absolutely no idea what Form A is saying, because they don't understand the language in which it is expressed.

Most students will have relatively few problems with Form B, because it tells them *what they can do*. "You can add the same number to both sides of an equation" gives the student something that they can *do*.

On the contrary, Form A is just a statement of *fact*. Most students don't recognize Form A as telling them what they can *do*. But it does. *Facts can tell you what to do, once you learn to make the correct translation.*

good reference material

Warren W. Esty of Montana State University has written a delightful text entitled **The Language of Mathematics**, that directly addresses student difficulties with the language of mathematics. It is highly recommended reading for students of mathematics at all levels.

how to translate Form A

So how is the student to translate Form A?

The first sentence says *Let a, b and c be real numbers.* This sentence tells the reader that the universal set for the variables a, b and c is $\mathbb{R}$: that is, until otherwise notified, when the reader sees the symbols a, b and c, they are assumed to represent real numbers.

CAUTION: Just because the symbol a is different from the symbol b does not mean that our choice for a must be different than the choice for b! The variable a can be any real number; the variable b can be any real number. If desired, we can choose both to be, say, 2.

Next comes a statement that two sentences are equivalent: $a = b$ is equivalent to $a + c = b + c$. This says that, *no matter what real numbers are represented by a, b and c*, the sentence $a = b$ will have the same truth value as the sentence $a + c = b + c$.

For example, suppose that we make the choices $a = 3$, $b = 3$ and $c = 4.2$. In this case the sentence '$a = b$' becomes '$3 = 3$', which is true. The sentence '$a + c = b + c$' becomes '$3 + 4.2 = 3 + 4.2$', which is also true.

As a second example, suppose that $a = 3$, $b = 2$ and $c = -1$. Then the sentence '$a = b$' becomes '$3 = 2$', which is false. The sentence '$a + c = b + c$' becomes '$3 + (-1) = 2 + (-1)$', which is also false.

Now we can rephrase the sentence:

$$a = b \quad \iff \quad a + c = b + c$$

(Note that this entire display is a mathematical sentence, which compares the 'smaller' mathematical sentences $a = b$ and $a + c = b + c$, telling us that they always have the same truth value, and hence can be used interchangeably.)

This FACT tells us that we never change the truth of an equation by adding the same number to both sides. In other words, adding the same number to both sides of an equation doesn't change its solution set.

subtraction is a special kind of addition

But what about subtraction? Does the sentence

$$a = b \iff a + c = b + c$$

also tell us that we can *subtract* the same number from both sides of an equation? Of course! Subtraction is just a special kind of addition:

$$x - y := x + (-y)$$

To subtract a number means to add its opposite. That is, to subtract y means to add $-y$. Notice that the left-hand side '$x - y$' of the sentence illustrates a pattern; the right-hand side '$x + (-y)$' tells us what to *do* with this pattern.

By using the language of mathematics, we get two for the price of one: *one* mathematical sentence has told us that both adding and subtracting the same number to (from) both sides of an equation doesn't change its solution set.

EXERCISE 4

♣ 1. What is a *theorem*?

♣ 2. Discuss the meaning of this mathematical theorem:
For all real numbers x, y and z:

$$x = y \iff x + z = y + z$$

If we choose $x = 3$, $y = 4$ and $z = 5$, what is this theorem telling us?

♣ 3. Does this 'theorem' make sense:
For all real numbers a, b and c:

$$x = y \iff x + z = y + z$$

Why or why not?

Continuing our translation of Form A, we come to the sentence:
If c is not equal to zero, then:

$$a = b \iff ac = bc$$

The first part of the sentence informs us that the universal set for c has changed: now, whenever the reader sees the variable c, is it assumed to be a *nonzero* real number. But as long as c is nonzero, the sentences '$a = b$' and '$ac = bc$' will always have the same truth values. In particular, they're both true at exactly the same times. Thus, multiplying both sides of an equation by a nonzero number won't change its solution set.

EXAMPLE

For example, take $a = b = 7$ and $c = -2$. Then the sentence '$a = b$' becomes '$7 = 7$', which is true. The sentence '$ac = bc$' becomes '$(7)(-2) = (7)(-2)$', which is also true.

If we take $a = 0$, $b = -7$ and $c = 3$, then the sentence '$a = b$' becomes '$0 = -7$', which is false. Also, '$ac = bc$' becomes '$(0)(3) = (-7)(3)$', which is also false.

division is a special kind of multiplication

But what about division? Providing $c \neq 0$, does the sentence

$$a = b \iff ac = bc$$

tell us that we can *divide* both sides of an equation by the same nonzero number? Of course! Division is just a special type of multiplication: for $y \neq 0$,

$$\frac{x}{y} := x \cdot \frac{1}{y}$$

To divide by y means to multiply by the reciprocal of y. Again, we get two for one.

EXERCISE 5

Consider this theorem:

For all real numbers a, b, and c:

$$a < b \iff a + c < b + c$$

♣ 1. What is this theorem telling you that you can *do*? Answer in English.

♣ 2. What is the theorem telling you when $a = 1$, $b = 2$ and $c = 3$?

♣ 3. What is the theorem telling you when $a = 2$, $b = 1$ and $c = 3$?

♣ 4. How might an algebra instructor 'translate' this theorem for the students?

EXERCISE 6

Consider this theorem:

For all real numbers a and b, and for $c < 0$:

$$a < b \iff ac > bc$$

♣ 1. What is this theorem telling you that you can *do*?

♣ 2. How might an algebra instructor 'translate' this theorem for the students?

♣ 3. Write a theorem, the way a mathematician would, that tells you that multiplying both sides of an inequality (using '<') by a positive number gives an equivalent inequality in the same direction.

EXAMPLE

The next example is *extremely important*. It illustrates the basic procedure used in solving a (simple) equation. It should seem trivial to you—but make sure you understand *why* you're doing what you're doing.

Problem: Solve the equation $-3x + 5 = 2$.

Solution: Find an *equivalent equation* that can be solved by inspection. That is, transform the original equation into an equivalent one of the form $x =$ *some number*.

$-3x + 5 = 2$	Begin with the original equation.
$(-3x + 5) - 5 = 2 - 5$	Isolate the x term by subtracting 5 from both sides. By the theorem, this does not change the solution set.
$\frac{-3x}{-3} = \frac{-3}{-3}$	Divide both sides by -3. By the theorem, this does not change the solution set.
$x = 1$	The result is an equivalent equation that can be solved by inspection.
	Since the solution sets of $x = 1$ and $-3x + 5 = 2$ are the same, the solution set of $-3x + 5 = 2$ is $\{1\}$.

$\iff$ *is implicit*

In practice, the solution to this problem would be written down as follows, merely as a list of equations:

$$\begin{aligned} -3x + 5 &= 2 \\ -3x &= -3 \\ x &= 1 \end{aligned}$$

Each line is a complete mathematical sentence. However, the sentences aren't put together into a cohesive paragraph. The reader is left to *guess* what the connection is between, say, $-3x = -3$ and $x = 1$ (they are equivalent).

NOTATION

$\iff$ *is explicit*

A much better way to write down the solution is:

$$-3x + 5 = 2 \iff -3x = -3 \iff x = 1$$

or

$$\begin{aligned} -3x + 5 = 2 \quad &\iff \quad -3x = -3 \\ &\iff \quad x = 1 \end{aligned}$$

Now, the relationship between the component sentences is clear: they are equivalent. Since we know what makes $x = 1$ true, we also know what makes $-3x + 5 = 2$ true. Here, the sentences have been combined into a cohesive mathematical paragraph.

The latter form

$$\begin{aligned} -3x + 5 = 2 \quad &\iff \quad -3x = -3 \\ &\iff \quad x = 1 \end{aligned}$$

is particularly nice, because the original equation $-3x+5 = 2$ stands out at the top of the left-hand column, and the much simpler equivalent equation $x = 1$ stands out at the bottom of the right-hand column.

This form can be 'annotated' easily as follows:

$$\begin{aligned} -3x + 5 = 2 \quad &\iff \quad -3x = -3 \qquad \text{(subtract 5)} \\ &\iff \quad x = 1 \qquad\qquad \text{(divide by } -3\text{)} \qquad (*) \end{aligned}$$

YOU WILL BE EXPECTED TO WRITE COMPLETE AND CORRECT MATHEMATICAL PARAGRAPHS IN THIS COURSE.

EXERCISE 7

♣ Solve the equation $3x - 7 = 1$. Be sure to write a complete mathematical paragraph. Tell what you're doing in each step of your solution. Use the form illustrated in (*).

EXERCISE 8

♣ Solve the inequality $3x - 7 < 1$. Be sure to write a complete mathematical paragraph. Tell what you're doing in each step of your solution. Use the form illustrated in (*).

INCORRECT notation

It is *incorrect* and completely unacceptable to write:

$$\begin{aligned} -3x + 5 = 2 \quad &= \quad -3x = -3 \\ &= \quad x = -1 \end{aligned}$$

Taken literally, this says that 2 is equal to -3 which is equal to -1. Absurd.

Remember:

- The equals sign '=' is used to compare *expressions*.
- The 'is equivalent to' sign ' $\iff$ ' is used to compare *sentences*.

What goes wrong if c is zero?

Note that multiplying by zero can change the truth value of a sentence. For example, the equation $3 = 5$ is false, but the equation $3 \cdot 0 = 5 \cdot 0$ is true. Therefore, *multiplying both sides of an equation by zero does not necessarily yield an equivalent equation*, and therefore is not allowed.

EXAMPLE
adding a solution

For example, consider the equation $x = 2$. It has solution set $\{2\}$. Multiplying both sides by x yields the new equation $x^2 = 2x$, which has solution set $\{0, 2\}$. Thus, the equations

$$x = 2 \quad \text{and} \quad x^2 = 2x$$

are *NOT* equivalent, since they have different solution sets.

What happened? Well, as long as x is nonzero, multiplication by x is just multiplication by a nonzero number, which doesn't alter the solution set. But when x is zero, multiplication by x took us from the false statement $0 = 2$ to the true statement $0 = 0$.

Thus, *multiplying by a variable expression may ADD a solution.* Adding a solution isn't really too serious, providing that you check your 'potential' solutions into the original equation at the final step.

EXAMPLE
losing a solution

More serious is this next situation. Begin with $x^2 = 2x$ and divide both sides by x, yielding the new equation $x = 2$. The equations are not equivalent and a solution has been *lost*. If we separately investigate the situation when x is zero, it will be discovered that 0 is also a solution of the original equation. If we neglect to do this, the solution 0 is lost forever.

EXERCISE 9
contradiction

♣ Solve the equation $3(x + 2) = (3x + 1) + 4$. Show that this equation is equivalent to the equation $6 = 5$, which is never true. This type of equation, which is always false, is sometimes referred to as a *contradiction*.

EXERCISE 10
identity

♣ Solve the equation $2x - (7 - x) = x + 1 - 2(4 - x)$. Show that this equation is equivalent to the equation $0 = 0$, which is true for all values of x. This type of equation is sometimes referred to as an *identity*.

QUICK QUIZ
sample questions

1. T or F: the sentences '$x = 2$' and '$x^2 = 4$' are equivalent. Justify your answer.
2. Write a theorem, the way a mathematician would, that says that adding the same number to both sides of an equation does not change its solution set.
3. Write a theorem, the way a mathematician would, that says that adding the same number to both sides of an inequality (using '>') does not change its solution set.
4. What is the 'implied domain' of the sentence '$\frac{1}{(x-3)y} = 2$'? Use correct set notation for your answer.
5. Fill in the blank: the goal in solving an equation is to transform it into an ________________ one that can be easily solved.
6. Fill in the blanks:

   The '=' sign is used to compare ________________.

   The '$\iff$' sign is used to compare ________________.

KEYWORDS
for this section

Equivalent sentences, implied domain for a sentence, solving by inspection, general approach for solving an equation, subtraction is a special kind of addition, division is a special kind of multiplication, multiplying by a variable, dividing by a variable, contradiction, identity.

You should be familiar with the notation $\iff$ and be able to use it correctly when solving equations and inequalities.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE (T), FALSE (F), or CONDITIONAL (C). The first one is done for you.

1. $x = 3 \iff 2x - 6 = 0$; SEN, T. Both sentences have the same implied domain, and the same solution sets.
2. $x = 4$
3. $x = 4 \iff 4 - x = 0$
4. $\frac{1}{x} = 3 \iff \frac{1}{3x} = 1$
5. $|y| = 0 \iff y = 0$
6. $|y| = 1 \iff y = 1$
7. y^3
8. $y^3 = 8 \iff y = 2$
9. $y^3 + 8 = 0 \iff y + 2 = 0$
10. $x(x-1) = 0 \iff x - 1 = 0$

♣ Using the theorems in this section, solve the following equations/inequalities. BE SURE TO WRITE COMPLETE AND CORRECT MATHEMATICAL SENTENCES.

11. $5x - 7 = 3$
12. $5 - 3y = 9$
13. $3x < x - 11$
14. $3t + 7 \geq -2$