1.3 Sets and Set Notation

Introduction

In algebra, the word *set* appears when discussing the *solution set* of an equation. There are many other places where sets are important. For example, sets will be used to simplify our discussion of graphing equations and functions. The focus of this section is sets, and the notation used in connection with sets.

DEFINITION
sets
well-defined

A *set* is a well-defined collection of objects. *Well-defined* means that, given any object, either the object *is* in the set, or *isn't* in the set.

EXAMPLE
a non-set

"*The collection of some people*" is not a set. It is not well-defined. To see this, observe that we cannot definitively answer the question: Is 'Carol' in this collection?

EXAMPLE
a set

"*The collection of all irrational numbers with the digit* 5 *in the* 10^{-2013} *slot of their decimal expansion*" is a set. Call it S. Given any number, either it is in S or it isn't. For example, 3 isn't in S, because 3 isn't irrational. Is π in S? This author doesn't know. But either it *does* have a digit 5 in the appropriate slot, or it doesn't. No other choices are possible.

Thus, to qualify as a set, one need only be certain that *any* object is either in the collection, or not. It's not necessary to know which of these two situations occurs.

NOTATION
$\in$, $\notin$

The sentence $x \in S$ means x *is an element of* S. It can also be read as:

$$x \text{ is a member of } S$$
$$x \text{ belongs to } S$$
$$x \text{ is in } S$$

The sentence $x \notin S$ means x *is NOT an element of* S.

♣ What is the 'verb' in the sentence $x \in S$?

NOTATION
set notation
roster (list) method

The members of a set are often separated by commas, and enclosed in braces $\{ \}$. That is, the elements are listed; this is called the *roster* or *list* method.

If the set contains many elements, then it is often convenient to use *dots* to continue an established pattern. This is illustrated by the following examples:

EXAMPLE

- The set $\{a, b, c\}$ contains 3 elements, a, b, and c. Roster notation is particularly useful when a set contains a small finite number of elements.
- The set of *counting numbers* is $\{1, 2, 3, \dots\}$. The dots indicate that the established pattern continues ad infinitum.
- The set $\{1, 2, 3, \dots, 100\}$ enumerates the counting numbers between 1 and 100, inclusive.
- Let $S = \{\ 1,\ \{1\},\ \{1, \{1\}\}\ \}$. Here, S is a set containing (among other things) sets. There are three elements in S: 1, $\{1\}$, and $\{1, \{1\}\}$. Thus it is correct to say: $1 \in S$, $\{1\} \in S$, and $\{1, \{1\}\} \in S$.

EXERCISE 1

♣ Let $S = \{2,\ \pi,\ \{2, \pi\},\ \{2\}\}$. How many elements does S have? What are they?

NOTATION
set-builder notation

Even more important for large (usually infinite) sets is the following *set-builder notation*:

Let $\mathcal{U}$ denote a universal set. The notation

$$\{x \in \mathcal{U} \mid \text{some property that } x \text{ is to satisfy}\}$$

is *extremely* useful in many cases where roster notation fails. Here, the vertical bar ‘ | ’ is read as *such that* or *with the property that*. The set includes all elements from the universal set that satisfy the stated property.

If the universal set is understood, one can more simply write

$$\{x \mid \text{some property that } x \text{ is to satisfy}\} \ .$$

EXAMPLE

For example,

$$\{x \mid x \text{ is a counting number and } x \geq 4\}$$

is read as *the set of all x such that x is a counting number and x is greater than or equal to* 4.

Another way to denote this set would be $\{4, 5, 6, \dots\}$. Yet another way would be $\{y \mid y \text{ is an integer and } y \geq 4\}$. Thus, we see that a given set can be expressed in different ways.

EXERCISE 2

♣ 1. Describe the set $\{-3, -2, -1, 0, 1, 2, 3\}$ in two different ways.

♣ 2. Describe the set $\{\dots, -1, 0, 1, 2\}$ in two different ways.

EXAMPLE

The set $\{N \mid N \text{ is a name that begins with C}\}$ is read as *the set of all N with the property that N is a name that begins with C*. Here, the universal set is understood to be the set of all possible names. Thus, the name *Carol* is an element of this set, but *Karol* is not. Observe that it would be extremely difficult to describe this set without set-builder notation.

NOTATION
interval notation
(,); *endpoints are not included*
[,]; *endpoints are included*

Interval notation is a particularly convenient way to denote intervals of real numbers.

Recall that the symbol := means *equals, by definition*. Define

$$\begin{aligned}
(a, b) &:= \{x \in \mathbb{R} \mid a < x < b\} \\
[a, b) &:= \{x \in \mathbb{R} \mid a \leq x < b\} \\
(a, b] &:= \{x \in \mathbb{R} \mid a < x \leq b\} \\
(a, \infty) &:= \{x \in \mathbb{R} \mid x > a\} \\
(-\infty, b] &:= \{x \in \mathbb{R} \mid x \leq b\}
\end{aligned}$$

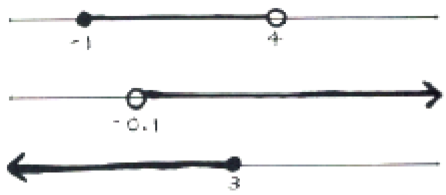
Other combinations are possible. Compound inequalities like $a < x < b$ are investigated in the Algebra Review at the end of this section, and the mathematical word ‘and’ is introduced.

Note how convenient set-builder notation is for these definitions. Observe that:

- *parentheses* (,) are used when an endpoint is *not* to be included
- *brackets* [,] are used when an endpoint *is* to be included
- The symbol ∞ is always used with parentheses. This is because ∞ is not a real number. It's more of an idea: given *any* real number, another real number can always be found that is greater.

EXERCISE 3

♣ Use interval notation to describe the sets shown below. (A solid dot indicates that an endpoint is included; a hollow dot indicates that an endpoint is excluded.)



(a, b) has two different meanings

The careful reader will observe that the notation (a, b) can be used to denote an *interval*, or an *ordered pair*. Context will determine which interpretation is correct.

NOTATION

empty set

$\emptyset$

There is exactly one set containing no elements. It is called the *empty set*, and is denoted by either { } or $\emptyset$.

Computer scientists use the symbol $\emptyset$ for the number zero, to distinguish it from the capital letter 'oh'. So if you are communicating with a computer scientist, it is probably better to use { } to denote the empty set.

capital letters are used to denote sets

Capital letters, like A, $\mathcal{B}$, S and Γ, are commonly used to denote sets.

Now, we are ready to define the *solution set* of an equation.

DEFINITION

solution set of an equation

solving an equation

The *solution set* of an equation is the set of all its solutions. To *solve an equation* means to find its solution set (i.e., find all solutions.)

EXAMPLE

solution sets

the quadratic formula

- The solution set of $x^2 = 4$ is $\{2, -2\}$.
- The solution set of $ax^2 + bx + c = 0$ for $a \neq 0$ is

$$\left\{ \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right\} .$$

  You learned this when you studied the *quadratic formula* in algebra.

  ♣ What is (are) the variable(s) in the equation $ax^2 + bx + c = 0$? Judging by the solution set given above, what is the universal set?

- The solution set of $x - 1 = 0$ is $\{1\}$.
- The solution set of $x = 0$ is $\{0\}$.

EXERCISE 4

♣ 1. Based on the definition of the *solution set of an equation*, write a precise definition for the *solution set of an inequality*.

♣ 2. Solve the following simple inequalities in one variable. Where possible, use interval notation for the solution sets.

a) $x > 4$

b) $y^2 \geq 0$

c) $t^2 < 0$

d) $|x| \leq 1$

♣ 3. How many solutions do inequalities usually have?

Now comes the $100 question: How do we go about *finding* the solution set of a given equation (or inequality)? This is the topic of the next section.

the mathematical word 'and'

Be sure to read the algebra review that follows, since the precise meaning of the mathematical word 'and' is introduced.

ALGEBRA REVIEW

compound inequalities, mathematical word 'and', integers, rational numbers

compound inequalities

A sentence like $a < x < b$ that uses more than one inequality symbol is called a *compound inequality*.

The compound inequality $a < x < b$ is really just a shorthand for two simple inequalities, connected by the mathematical word 'AND':

$$a < x \;\text{ AND }\; x < b\ .$$

Thus, to truly understand compound inequalities, one must understand the mathematical word 'and'.

the mathematical word 'and'

Let P and Q denote mathematical sentences that are either true or false. For example, P might be the true sentence '$3 > 1$'. Similarly, Q could be the false sentence '$5 < 5$'. The mathematical word 'AND' gives a way to *combine* the sentences P and Q into a 'bigger' mathematical sentence.

By definition, a mathematical sentence of the form

$$P \;\text{ AND }\; Q$$

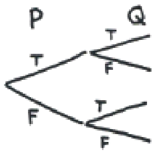
is true exactly when P is true and Q is true.

For example, the mathematical sentence '$1 < 3$ and $3 < 5$' is true, since both $1 < 3$ and $3 < 5$ are true.

However, the sentence '$3 < 3$ and $3 < 5$' is false, because $3 < 3$ is false.

truth table for 'P and Q'

There is a more precise way to discuss the mathematical sentence 'P and Q'. Note that the truth value (true or false) of this sentence depends on the truth values of P and Q. P can be true or false. Q can be true or false. Together, there are four possible combinations of truth values, which are summarized in the truth table below.



P	Q	P AND Q
T	T	T
T	F	F
F	T	F
F	F	F

This truth table shows that:

- When P is true and Q is true, the sentence 'P and Q' is true.
- When P is true and Q is false, the sentence 'P and Q' is false.
- When P is false and Q is true, the sentence 'P and Q' is false.
- When P is false and Q is false, the sentence 'P and Q' is false.

EXERCISE 5

Determine the truth value (T or F) of the following sentences:

♣ 1. $\pi > 3$ and $|2| = 2$

♣ 2. $|3 - \pi| > 0$ and $-3^2 = -9$

♣ 3. $3 \le 3$ and $-2 < -4$

♣ 4. $1 \in (1, 3)$ and $1 \in [1, 3)$

♣ 5. $1 \in \{x \mid x < 1\}$ and $1 \in \{\,\{1\}\,\}$

Determine the value(s) of x for which each of the following sentences is true; false. Where possible, use interval notation to express your answer.

♣ 6. $x > 0$ and $x > 2$

♣ 7. $x > 0$ and $x < 2$

♣ 8. $x > 0$ and $x < -2$

return to compound inequalities

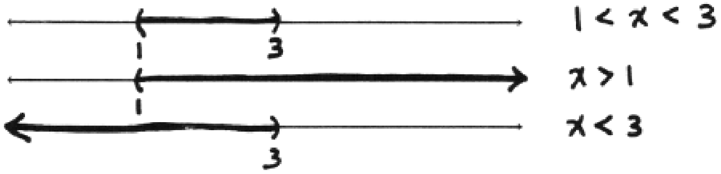
With the precise meaning of 'and' now in hand, we can return to a study of compound inequalities.

For what values of x will the mathematical sentence

$$a < x \text{ AND } x < b$$

be true? Only for values of x for which both $a < x$ $(x > a)$ is true, and $x < b$ is true. That is, only for the values of x which are greater than a, and (at the same time) less than b.

The sketches below illustrate this construction for the compound inequality $1 < x < 3$.



EXERCISE 6

♣ 1. Make a similar sketch to explain the compound inequality

$$2 \ \le \ x \ < \ 5 \ .$$

♣ 2. Discuss the meaning of the compound inequality $3 < x < 1$. Are there any choices for x that make this true? Does it make sense to write $a < x < b$ if a is greater than b?

EXERCISE 7
$\mathbb{Z}$
integers

The symbol $\mathbb{Z}$ is frequently used to denote the set of integers, so we can write

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

The German word for 'numbers' is 'zahlen', which could explain the choice of the letter $\mathbb{Z}$.

♣ What are the positive integers? negative? nonnegative? nonpositive? Answer using correct set notation. (Hint: The word *nonnegative* means *not negative*. Thus, using interval notation, the nonnegative real numbers are $[0, \infty)$. Figure out what *nonpositive* means.)

EXERCISE 8
$\mathbb{Q}$ *(for 'Quotient')*
rational numbers

The symbol $\mathbb{Q}$ is frequently used to denote the set of rational numbers (since they are *Quotients* of integers). By long division, an equivalent characterization of $\mathbb{Q}$ is the set of all real numbers with finite or infinite repeating decimal representations. For example, $\frac{1}{7} = 0.\overline{142857}$ and $\frac{2}{5} = 0.4$.

♣ 1. Using long division, find the decimal representations of $\frac{1}{6}$, $\frac{2}{7}$ and $\frac{1}{25}$.

A rational number is in *reduced form* if there are no factors common to both numerator and denominator. For example, $\frac{6}{8}$ is not in reduced form, since

$$\frac{6}{8} = \frac{2 \cdot 3}{2 \cdot 4} = \frac{3}{4} \ .$$

A rational number in reduced form can be expressed as a *finite* decimal only if the denominator has no factors other than 2's and 5's. For example, $\frac{7}{400}$ has a finite expansion since

$$\frac{7}{400} = \frac{7}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5} = \frac{7}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5} \cdot \frac{5 \cdot 5}{5 \cdot 5} = \frac{175}{10^4} = 0.0175 \ .$$

♣ 2. Why is it that fractions with only 2's and 5's downstairs can be written as finite decimals? (Study the previous example.)

♣ 3. Decide (without using your calculator) if $\frac{3}{120}$ has a finite decimal expansion.

♣ 4. Decide if $\frac{41}{333}$ has a finite decimal expansion.

♣ 5. Decide if $\frac{10}{81}$ has a finite decimal expansion.

QUICK QUIZ
sample questions

1. T or F: the set $\{\,1\,,\,2\,,\,\{3,4\}\,\}$ has 4 members.
2. T or F: $1 \in (1,3)$.
3. T or F: $3 \in \{t \in \mathbb{R} \mid 2 < t < 5\}$
4. T or F: $3 > 3$ and $3 \le 3$
5. T or F: $\frac{3}{105}$ has a finite decimal expansion.

KEYWORDS
for this section

Sets, well-defined, roster method, set-builder notation, interval notation, empty set, solution set of an equation, solving an equation, quadratic formula, nonnegative, nonpositive, compound inequality, mathematical word 'and'.

You should know the symbols $\in$, $\notin$, $\{\ \}$, $\emptyset$, $:=$, $\mathbb{Z}$, and $\mathbb{Q}$. You should know what types of letters are commonly used to denote sets.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE (T), FALSE (F), or CONDITIONAL (C).

1. $\{3\}$
2. $\{1,2,3\}$
3. $1 \in \{1,2,3\}$
4. $0 \in [0,\frac{1}{2})$
5. $0 \in (0,\frac{1}{2})$
6. $\frac{1}{2} \in [0,\frac{5}{9})$
7. $x \in S$
8. $1 \notin \{1,2,3\}$
9. $x \in \{1,2,3\}$
10. $1 \in \{t \,|\, t \le 1\}$
11. $\{x \,|\, x \ge 1\}$
12. $1 \in \{\ \{1\}\ ,\ \{1,\{1\}\}\ \}$
13. $\{1\} \in \{\ \{1\}\ ,\ \{1,\{1\}\}\ \}$
14. $x > 1$ and $x < 1$
15. $y \ge 1$ and $y \le 1$
16. $x \le 3$ and $x > 5$
17. $|x| \ge 0$ and $x^2 \ge 0$
18. $|x| > 0$ and $x^2 \ge 0$
19. The set $\{\ \{1\}\ ,\ \{1,\{2\}\}\ \}$ has two elements.
20. The set $\{\ \{a\}\ ,\ \{b,c\}\ \}$ has three elements.
21. The number $\frac{3}{7}$ has a finite decimal expansion.
22. The number $\frac{7}{35}$ has a finite decimal expansion.