1.2 The Role of Variables

variables

In this section, a name is given to mathematical sentences that are 'sometimes true, sometimes false'—they are called *conditional* sentences. The truth of such sentences depends on the choices that are made for the objects that are allowed to *vary* in the sentence. In mathematics, an object that is allowed to *vary* is appropriately called a *variable.* Variables play a very important role in mathematics, and are the focus of the current section.

sentences come in several flavors

As discussed in the previous section, sentences come in several flavors. There are:

true

TRUE sentences, such as:

- $\pi + (-\pi) = 0$,
- $t^2 \geq 0$,
- $(-3)^2 = 9$, and
- $x + x + x = 3x$.

Note that the sentence $x+x+x = 3x$ is true for *any* real number x. A sentence with verb '=' is called an *equation.*

false

FALSE sentences, such as:

- $5^0 = 5$,
- $\sqrt{(-7)^2} = -7$,
- $(3+2)^2 = 3^2 + 2^2$, and
- $x^2 < 0$

Note that the sentence $x^2 < 0$ is false for *all* real numbers x. A sentence with verb $<$, $\leq$, $>$, or $\leq$ is called an *inequality.* The prefix 'in' is commonly used to mean 'not'; as in the English words *inept*, *insane*, and *insecure.* Hence, **in***equality* means, roughly, *NOT equal.*

Mathematicians *hate* to see false sentences written down, except perhaps in a book on logic!

conditional

CONDITIONAL sentences, such as:

- $x = 3$,
- $x^2 + 2x + 1 > 0$,
- $\sqrt{2x+1} \neq 5$, and
- $y = 2x + 4$.

This is a very interesting type of sentence. By definition, a *conditional sentence* is one that is sometimes true and sometimes false.

For example, the equation $x = 3$ is true when x is 3 and false when x is not 3. Thus, the *condition* of the sentence depends on the value(s) of the variable(s) involved.

EXERCISE 1
classifying sentences

♣ Classify the following equations as (always) true, (always) false, or conditional.

♣ For those that are conditional, can you say when they are true? False?

1. $2 \cdot 4^2 = 32$ (See Algebra Review—exponents.)
2. $2x^3 = 2 \cdot x \cdot x \cdot x$
3. $x - 3 = 0$
4. $\sqrt{9} = -3$
5. $x + y = 4$
6. $\sqrt{(-9)^2} = -9$
7. $|x| > 0$

'place holders'

When working with conditional sentences, the concept of 'place holder' becomes important. To illustrate this point, consider a familiar example.

solving an equation

Recall from algebra that to *solve an equation* like

$$2x + 4 = 10$$

means to find the number that makes it true. (You learned in algebra that equations of this type have only one solution.) To accomplish this, a number (here denoted by 'x') must be found that has the following property: when it is doubled (multiplied by 2), and 4 is added to it, the result is 10.

Instead of pulling out 'standard' algebra techniques right now, just stop and think. First, what number, when added to 4, yields 10? The number 6. Thus, twice the desired number x must equal 6. Therefore, x must equal 3.

Now if you are asked to solve the equation $2t + 4 = 10$, you should recognize that it has already been done. The only difference is that the letter 't' is used as a place holder, instead of 'x'.

EXERCISE 2
solving simple equations mentally

♣ 1. Without writing anything down, solve the equation $2x - 8 = 6$.

♣ 2. Now solve the equation $(x - 2)^2 = 9$ mentally.

The notion of 'place holder' is much too imprecise for the language of mathematics. To formalize this notion, the concept of *variable* is introduced.

The word *set* appears in the next definition. For now, just think of a *set* as a collection of things (like numbers). Sets will be discussed in more detail in the next section.

DEFINITION
variable
universal set

A *variable* is a symbol (often a letter) that is used to represent a member of a specified set.

This 'specified set' is referred to by mathematicians as the *universal set*.

Thus, the *universal set* gives the objects (often numbers) that we are allowed to draw on for a particular variable.

symbols traditionally used to denote variables

The letters x, y, and t are commonly used in elementary mathematics courses to denote variables, with universal set $\mathbb{R}$.

the role of definitions in mathematics

Definitions are extremely important in mathematics. In order to communicate effectively, people must agree on the meanings of certains words and phrases. English occasionally fails in this respect. Consider the following conversation in a car at a noisy intersection:

Carol: "Turn left!"

Bob: "I didn't hear you! Left?"

Carol: "Right!"

Question: Which way will Bob turn? It depends on how Bob interprets the word 'right'. If he interprets it as the opposite of 'left', he will turn right. If he interprets it as 'correct', he will turn left.

Such ambiguity is not tolerated in mathematics. By *defining* words and phrases, mathematicians assure that everyone agrees on their meaning.

EXERCISE 3

ambiguity in English

♣ Come up with another example of an English word or phrase that is ambiguous, and where this could cause communication problems.

EXAMPLE

same equation, different universal sets

To illustrate the roles that variables and universal sets play in solving equations, consider the equation

$$x^2 = -1 \ .$$

You are asked to solve this equation. There is only one variable, x. What is the universal set? You had better find out, because it will affect your answer.

If the universal set is $\mathbb{R}$, then you must find all *real numbers* which, when squared, equal -1. There are none. So in this case, you would say that there are no solutions to the equation.

Suppose, however, that the universal set is $\mathbb{C}$ (the complex numbers). (See Algebra Review—complex numbers.) In $\mathbb{C}$, there are *two* numbers that make the equation $x^2 = -1$ true; i and $-i$. So in this case two solutions are obtained.

EXERCISE 4

seeing how the universal set affects the solutions of an equation

♣ Solve the equation $x^4 = 1$.

1. First, let $\mathbb{R}$ be the universal set. How many solutions do you get?
2. Next, let $\mathbb{C}$ be the universal set. How many solutions do you get?
3. Finally, let the integers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ be the universal set. How many solutions do you get? (See Algebra Review—integers.)

EXERCISE 5

♣ Solve the equation $x^2 = 2$ taking the universal set to be:

1. $\mathbb{R}$
2. the rational numbers (See Algebra Review—rational and irrational numbers.)
3. the irrational numbers
4. the integers

EXERCISE 6
theorem
the Fundamental Theorem of Algebra

In mathematics, the word *theorem* is used to denote a result that is both *true* and *important*.

♣ Look up the *Fundamental Theorem of Algebra* in an algebra book. What is it saying? In particular, it tells us that the complex numbers $\mathbb{C}$ are 'nicer' than the real numbers $\mathbb{R}$ in one way. What way is this?

three main uses of variables

Variables are frequently used:

- **in mathematical expressions to denote quantities that are allowed to change (vary)**

For example, let A denote the area of a circle that has radius r. Then, $A = \pi r^2$. Here, r and A are variables, because they vary from circle to circle. The symbol π, on the other hand, is *not* a variable. It *never* changes. It is called a *constant*.

- **in equations and inequalities to denote a quantity that is initially unknown, but that one would like to know**

For example, in the equation $2x - 3 = 1$, the value of x that makes this true is (initially) unknown. One goal in algebra is to find the appropriate number.

- **to state a general principle**

For example, the sentence

$$\text{For all real numbers } x \text{ and } y,\ x + y = y + x.$$

should be recognized as a precise statement of the *Commutative Law Of Addition*. To 'commute' means to change places; say, to go from home to work and then back again. All commutative laws have the same theme: a 'changing of places' of the objects involved does not affect the final answer.

DEFINITION
constant

A *constant* is a quantity that *does not vary* (that is, remains the same—constant) during some discourse.

symbols traditionally used to denote constants

All specific real numbers are constants: like 5, 0, $\sqrt{2}$ and π.

Symbols are frequently used to denote constants. This can be confusing, because symbols are also used to denote variables. However, there are some strong mathematical *conventions* that exist when naming constants. No reasonable person would use x or y to denote a constant, because these letters are too often used to denote variables.

Frequently, the 'earlier' letters in alphabets, like a, A, α or b, B, β or c, C, γ are used to denote constants.

EXAMPLE
types of equations

$2x + \sqrt{x} - 5 = 0$ is an equation in one variable, x.

$x^2 - x = \sqrt{3}y - 7$ is an equation in two variables, x and y.

$x + \frac{y}{z+x+y} = 4$ is an equation in three variables, x, y, and z.

variables vs. constants

The equation $Ax + By + C = 0$, where A and B are not both zero, is an equation in two variables, x and y. Here, convention dictates that A, B and C are *constants*, not variables. If we genuinely wanted A, B and C to be treated as variables, this would need to be stated explicitly.

The equations that can be expressed in this way $Ax + By + C = 0$ form a very important class of equations called the *linear equations in two variables*. There is one equation for each choice of the constants A, B, and C. For example, $2x+3y-5 = 0$ is a linear equation in the two variables x and y. So is $x-y = \pi$, since it can be written as $x - y - \pi = 0$.

EXERCISE 7

♣ 1. Give an example of a linear equation in the variables w and z.

♣ 2. Give an example of an inequality in the variable x, where this variable appears three times.

♣ 3. A book defines a *quadratic equation* as one that can be written in the form

$$\alpha x^2 + \beta x + \gamma = 0 \ , \quad \alpha \neq 0 \ .$$

According to the conventions in mathematics, what should you assume are variables? What are constants?

solving equations

Now that variables, constants, and equations have been discussed, it's time to talk more precisely about *the solutions of equations*.

DEFINITION

solution of an equation in one variable

satisfy

A *solution of an equation in one variable* is a number (from the universal set) which, when substituted for the variable, makes the equation into a true statement. Such a number is said to *satisfy* the equation.

In this course, the universal set is $\mathbb{R}$, unless otherwise specified. Thus, we will be looking only for REAL NUMBER solutions.

EXAMPLE

- The number 3 is a solution of the equation $x = 3$. Are there any others?
- The numbers 2 and -2 both satisfy the equation $y^2 = 4$. Are there any others?
- The number $\sqrt{3}$ is a solution of the equation $t^2 + 2 = 5$. Are there any others? (Yes, $-\sqrt{3}$.)
- The equation $x^2 = -4$ has no solutions (in the real numbers).

DEFINITION

solution of an equation in two variables

A *solution of an equation in two variables* is a *pair* of numbers which, when substituted for the variables, makes the equation into a true statement.

EXAMPLE

The choices $x = 2$ and $y = 2$ give a (single) solution of the equation $x + y = 4$. Note that this choice of *two* numbers yields only *one* solution; it is incorrect to say '$x = 2$, $y = 2$ **are** solutions of $x + y = 4$'.

The pair $x = 1$, $y = 3$ is another solution. So is $x = 1.1$, $y = 2.9$. Any guesses as to how many solutions there are? (ANS: an infinite number!)

n-tuple
ordered pair

To talk about equations in 2 or more variables, the concept of *n-tuple* is used. An *n-tuple* is an ordered list of n numbers. By convention, the n numbers are separated by commas, and enclosed in parentheses (,).

For example, $(1,2)$ is a 2-tuple, more commonly known as an *ordered pair*.

The 5-tuple $(1,2,3,4,5)$ is different from the 5-tuple $(5,4,3,2,1)$ since the order is important.

DEFINITION
solution of an equation in n variables

A *solution of an equation in n variables* is an n-tuple of numbers which, when substituted for the n variables, makes the equation into a true statement.

A convenient way to denote a typical n-tuple is

$$(x_1, x_2, x_3, \ldots, x_n) .$$

Note that the subscript on the variable tells the position in the list. Thus, x_3 denotes the third number in this list.

EXERCISE 8

Consider the equation in four variables $x_1 + 2x_2 + 3x_3 + x_4 = 0$.

♣ 1. What are the four variables in this equation?

♣ 2. What is meant by *a solution* to this equation?

♣ 3. Find a solution to this equation (at least one solution should be obvious).

♣ 4. Find another solution. Any guesses as to how many solutions there are?

finding ALL the solutions

One is usually interested in finding *all* the solutions of a given equation. To discuss this collection of solutions precisely, the concept of *set* will first be reviewed. This is the subject of the next section.

ALGEBRA REVIEW

exponents, complex numbers, integers, rational & irrational numbers

strength of operations

Recall from algebra that the expression -3^2 means $(-1)\cdot(3^2)$. That is, take 3, square it, and *then* multiply by -1. This is a consequence of the mathematical *conventions* regarding order of operations. Mathematicians have agreed that when no order is specified (say with parentheses), then the *strongest* operations will act first. This agreement is usually referred to as the *order of operations*. But what is the 'strength' of the various operations?

addition and subtraction have equal 'strength'

Start with addition. Since subtraction is a special kind of addition,

$$a - b := a + (-b) \ ,$$

both addition and subtraction have equal strength. The symbol ':=' just used emphasizes that the equality is *by definition*.

multiplication is a sort of 'super-addition'

Multiplication is a sort of 'super-addition'. For example, $3\cdot 4 = 4+4+4 = 3+3+3+3$. Thus, multiplication is 'stronger than' addition. Since division is a special kind of multiplication,

$$\frac{a}{b} := a\cdot\frac{1}{b} \ ,$$

both multiplication and division have equal strength.

exponentiation is a sort of 'super-multiplication'

Exponentiation is a sort of 'super-multiplication'. For example, $2^3 = 2\cdot 2\cdot 2$. Thus, exponentiation is 'stronger than' multiplication.

A sentence that students sometimes use to help them remember the conventions about order of operations is:

Please **E**xcuse **M**y **D**ear **A**unt **S**ally.

'P' stands for **P**arentheses, 'E' for **E**xponents, 'M' and 'D' for **M**ultiplications and **D**ivisions, 'A' and 'S' for **A**dditions and **S**ubtractions.

exponents are short-sighted

With these conventions, the exponentiation must be done before the multiplication in the expression -3^2. One way to remember that -3^2 means $(-1)\cdot(3^2)$ is that *exponents are extremely short-sighted*. When the exponent 2 'looks down' in -3^2, all it 'sees' is a 3, so this is what gets squared. However, when the exponent 2 'looks down' in $(-3)^2$, it sees a group, and in that group is a -3. So this is what gets squared.

$x^0 = 1$ for $x \neq 0$

By definition, $x^0 = 1$ for all real numbers x except 0. The expression 0^0 is undefined, just as division by zero is undefined. Thus,

- $5^0 = 1$,
- $\pi^0 = 1$, and
- $(-\sqrt{7})^0 = 1$.

♣ What is -7^0? $(-7)^0$? x^0? (Careful!)

Why is 0^0 undefined?

Why is 0^0 undefined? Here's one reason. Think of every real number as corresponding to a drawer in a (very large) filing cabinet. Thus, 2, $5-3$, and $\frac{2\pi}{\pi}$ all go into the same drawer. There's also a drawer to accomodate things that are 'not defined', like $\frac{2}{0}$.

Mathematicians want the sentence

$$\frac{x^3}{x^3} = x^{3-3} = x^0$$

to *always be true.* That is, for *any* real number x, the names $\frac{x^3}{x^3}$, x^{3-3}, and x^0 should all go into the same 'drawer'. Since $\frac{x^3}{x^3}$ is not defined when x is zero, we also want x^0 to be undefined when x is zero.

The Freshman's Dream

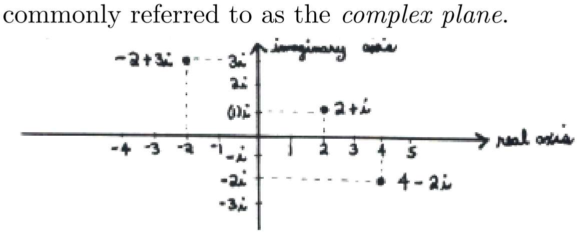
The 'Freshman's Dream' is that $(a+b)^2$ is equal to a^2+b^2. Unfortunately, this is *but* a dream. Take, for example, $a=1$ and $b=1$. Then $(a+b)^2 = (1+1)^2 = 4$ but $a^2+b^2 = 1^2+1^2 = 2$. The correct expression is

$$(a+b)^2 = (a+b)(a+b) = a^2+2ab+b^2\ .$$

the number $i := \sqrt{-1}$

If we are only allowed to use real numbers, the equation $x^2 = -1$ has no solution. (♣ Why not?) To accomodate situations such as this, the real numbers can be 'extended' to a larger number system, by defining *one* new number. This 'new' number is usually denoted by the letter i, and is defined by $i := \sqrt{-1}$. That is, i is a number which, when squared, equals -1.

By introducing this single new number i, and using the scheme illustrated below, we are given access to a whole *plane* of numbers, called the *complex numbers.* This plane is commonly referred to as the *complex plane.*



the complex numbers, $\mathbb{C}$

More precisely, the *complex numbers*, denoted by $\mathbb{C}$, are numbers that can be expressed in the form

$$a+bi\ ,$$

where a and b are real numbers, and $i := \sqrt{-1}$. Remember that the symbol ':=' is used to emphasize that this is the *definition* of i.

The complex numbers include all the real numbers (just take $b=0$), in addition to many numbers that are *not* real.

Electrical engineers use the complex numbers in studying current flow, and by so doing are able to eliminate a great deal of the drudgery involved in analyzing sinusoidal circuits. However, electrical engineers have to give $\sqrt{-1}$ a different name, because i is already used to denote current. So they define $j := \sqrt{-1}$.

integers

The *integers* are the numbers

$$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} .$$

The dots ‘ ... ’ indicate that the established pattern is to be repeated *ad infinitum*. The symbols i, j, k, m and n are often used to denote integers. For example, when a mathematician says *For all* $n > 1$, that mathematician is allowing the possibility that n is 2 or 100 or 2017, but not 5.2.

rational numbers

The *rational numbers* are those real numbers that can be expressed as a ratio of integers, with non-zero denominator. The word ‘ratio’ imbedded in ‘rational’ should help you remember this definition. Thus, $\frac{2}{3}$ is a rational number.

Is 5 rational? Certainly: it can be written as $5 = \frac{5}{1} = \frac{-10}{-2} = \ldots$. Note that the symbols 5, $\frac{5}{1}$ and $\frac{-10}{-2}$ all represent the unique real number 5, but the last two are more convenient names to use when determining that 5 is a rational number.

irrational numbers

The *irrational numbers* are those real numbers that are *not* rational. Thus, they cannot be expressed as a ratio of integers. The prefix ‘irr’ is commonly used in English to negate: consider *irregular*, *irreconcilable*, *irrelevant*, and *irresponsible*. You might recall from algebra that irrational numbers can alternately be described as the real numbers having infinite, non-repeating decimal expansions. So—we can’t write an irrational number as a ratio of integers, and we can’t write it in decimal form. How should we discuss it? Answer: give it a symbolic name (like π, or e, or $\sqrt{2}$)!

the irrational number π

The most familiar irrational number is π. The most common *approximations* to π are

$$\pi \approx \frac{22}{7} \quad \text{and} \quad \pi \approx 3.14159 .$$

Precisely, π is the ratio of the circumference to diameter of *any* circle. Here’s a great elementary school classroom exercise: have students bring in a circular object. Give them each a piece of string and a ruler. Have them measure the circumference of the circle and its diameter, then walk to the class computer and compute

$$\frac{\text{circumference}}{\text{diameter}} .$$

Within measuring error, each student will get a number close to 3.1!

QUICK QUIZ
sample questions

1. According to normal conventions in mathematics, what are the variables in the equation
$$Ax^2 + Bxy + Cy^2 = 0 \ ?$$
What are the constants?
2. Solve $x^2 = 3$ taking the universal set to be $\mathbb{R}$. Then, take the universal set to be the integers.
3. What does is mean to ‘solve’ an equation? (Answer in English.) Give three solutions of the equation $x + y = 4$.
4. Classify each of the sentences
$$x^2 \geq 0 \ , \quad x > 0$$
as (always) true, (always) false, or conditional.
5. List two of the three main uses for variables.

KEYWORDS *for this section*	*Conditional sentences, equation, inequality, variable, universal set, constant, complex numbers, integers, rational numbers, irrational numbers, solution of an equation in* 1, 2, *and* n *variables, satisfying an equation,* n*-tuple, definition, theorem.* You should know what letters are commonly used to denote variables and constants. You should also know what letters are commonly used to denote integers.

END-OF-SECTION EXERCISES

♣ Classify each entry in the list below as: an expression (EXP), or a sentence (SEN).

♣ Classify the truth value of any entry that is a *sentence*: (always) TRUE (T), (always) FALSE (F), or CONDITIONAL (C).

1. π
2. $\pi > 3$
3. $\pi = 3.14$
4. $i^2 + 1 = 0$
5. $\pi \approx 3.14$
6. A common rational approximation to π is $\frac{22}{7}$.
7. π is expressable as a ratio of integers.
8. The number i satisfies $y^2 = -1$.
9. The equation $3x^2 + 2x - \sqrt{x} = 0$ has three variables.
10. In the equation $ax^2 + bx + c = 0$, the variables are a, b, c and x.
11. In this course, the universal set is assumed to be the real numbers.
12. The 3-tuple $(1, 2, 3)$ is identical to the 3-tuple $(2, 1, 3)$.
13. $x^0 - 1$
14. $(y - z)^2 = y^2 - z^2$

♣ Solve each equation, taking the universal set to be:

a) $\mathbb{R}$

b) the rational numbers

c) the integers

15. $x^3 - 1 = 0$
16. $x^2 = 7$
17. $(x - 1)(x + \pi)(2x - 3) = 0$
18. $x(x^2 - 4)(x^2 - 2) = 0$
19. The Fundamental Theorem of Algebra tells us that the equation $x^3 = 1$ must have three solutions in $\mathbb{C}$.

    a) Plot the points 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ in the complex plane.

    b) Show that all these complex numbers lies on the circle of radius 1, centered at the origin. (Hint: use Pythagorean's Theorem.)

    c) Show that all these complex numbers satisfy the equation $x^3 = 1$.

    d) Solve $x^3 - 1 = 0$ with universal set $\mathbb{C}$.